Applicability of the Bi-material Notch Model to Analysis of Failure Initiation in Compound Structures

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Abstract—**The fracture mechanics analysis of bi-material notches is derived as the generalization of the classic linear elastic fracture mechanics approaches to general singular stress concentrators. The model of a bi-material notch is suitable to simulate a stress state of number of singular stress concentrators in bodies composed of two or more different materials. The paper outlines the possibilities of application of the bi-material notch model to analysis of crack initiation at the surface of solids composed of two materials.**

Index Terms—**Bi-material notch, crack initiation, singular stress concentrator.**

I. Introduction

THE model of a bi-material notch is suitable to simulate a number of construction points from which a failure is initiated. In the case of layered or fibre composite locations where the layers or fibres touch the surface of the composite body the singular stress concentrations occur. Such places are often responsible for crack initiation and consequently for the final failure of the whole body. Ignoring the stress concentrators can have critical consequences. On the other hand, considering them to be cracks is not correct either. The power of the singularity of the concentrators generally differs from ½ so the procedures of classic fracture mechanics cannot be used.

The fracture mechanics analysis of bi-material notches is derived as the generalization of the classic linear elastic fracture mechanics approaches to general singular stress concentrators. The basic assumptions of the study are small-scale yielding and a bi-material interface of a welded type, thus crack initiation is not considered along the interface, but into one of the materials. The model of a bi-material notch allows to simulate the stress state of many of the singular stress concentrators in bodies composed of two different materials.

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This contribution outlines the possibilities of application of the bi-material notch model to analysis of crack initiation at the surface of solids composed of two materials.

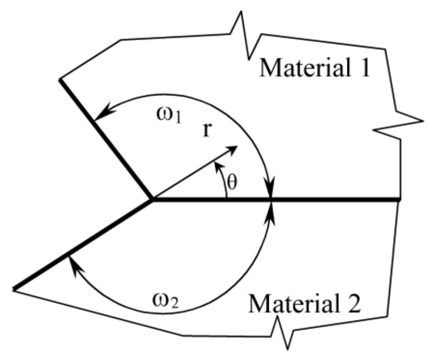

Fig.1 A bi-material notch characterized by the angles ω_1, ω_2, material properties of materials 1 and 2, and its polar coordinate system r, θ.

The general configuration of the bi-material notch is shown in fig.1. The geometry of the notch is given by the angles ω_1, ω_2 and the material properties characterised by the elastic constants E_1, ν_1, E_2, ν_2 corresponding to the materials 1 and 2. In the previous articles [1], [2] we presented procedures suitable for estimation of crack initiation conditions in the vicinity of bi-material notches. Now the application possibilities are discussed.

II. Bi-material Notch Model

A. Stress Distribution

The singular stress distribution is derived on the basis of Airy stress functions in the form of Williams` expansion [3]. In most of the geometrical and material configurations of a bi-material notch there are two terms of the expansion with the real stress singularity exponents p_1 and p_2 in the interval (0; 1). Contrary to a crack in a homogeneous material, the exponents differ from ½ and, furthermore, each singular term includes both normal and shear mode of loading, see [2] for detail. Then the singular stress components can be written in the polar coordinates:

$$\sigma_{mrr} = \sum_{k=1}^{2} \frac{H_k}{\sqrt{2\pi}} r^{-p_k} F_{rrkm} \quad (1a)$$

$$\sigma_{m\theta\theta} = \sum_{k=1}^{2} \frac{H_k}{\sqrt{2\pi}} r^{-p_k} F_{\theta\theta km} \tag{1b}$$

$$\sigma_{mr\theta} = \sum_{k=1}^{2} \frac{H_k}{\sqrt{2\pi}} r^{-p_k} F_{r\theta km} \ , \tag{1c}$$

where

$$F_{rrkm} = (2-p_k)(-a_{mk}\sin((2-p_k)\theta) - b_{mk}\cos((2-p_k)\theta) +$$
$$+3c_{mk}\sin(-p_k\theta) + 3d_{mk}\cos(-p_k\theta))$$
$$F_{\theta\theta km} = (p_k^2 - 3p_k + 2)(a_{mk}\sin((2-p_k)\theta) +$$
$$+b_{mk}\cos((2-p_k)\theta) + c_{mk}\sin(-p_k\theta) + d_{mk}\cos(-p_k\theta))$$
$$F_{r\theta km} = (2-p_k)(-a_{mk}\cos((2-p_k)\theta) +$$
$$+b_{mk}\sin((2-p_k)\theta) + c_{mk}\cos(-p_k\theta) - d_{mk}\sin(-p_k\theta))$$

The subscript m differentiates the materials 1 and 2 where the stresses are determined. The value H_k is the generalized stress intensity factor (GSIF), which follows from the numerical solution of the studied geometry with given materials and boundary conditions [4], [5], [2].

B. Criteria of crack initiation

As the structures modelled with the bi-material joint exhibit a singularity different from those of a crack (i.e. different from ½), the stability criteria originally suggested for a crack have to be generalized. Nevertheless, the new suggested criteria of the direction and stability depend on the fracture toughness K_{IC} (for brittle fracture) or the threshold value K_{Ith} (for fatigue failure) that are ascertained for a crack in homogeneous materials so they are commonly accessible. As mentioned above, both the criteria of the initial direction of crack propagation (from the bi-material notch tip) and the criteria of stability of the bi-material notch are suggested. Generally, the criteria are derived on the basis of several controlling variables (variables determining conditions of crack initiation) but here only the criteria of the mean value of the tangential stress are presented. These criteria are based on monitoring the tangential stress around the notch tip. The criterion suggested in [6] is here generalized to two singular terms occurring in the case of bi-material notches. The mean value of $\sigma_{\theta\theta}$ component over a certain distance d

$$\overline{\sigma_{\theta\theta}}(\theta) = \frac{1}{d}\int_0^d \sigma_{\theta\theta}(r,\theta)\mathrm{d}r \tag{2}$$

determined in dependence on the polar angle θ is considered as the controlling variable. By analogy with cracks in the homogeneous case it is supposed that the crack at the bi-material notch tip is initiated in the direction θ_0 where $\overline{\sigma_{\theta\theta}}(\theta)$ has its maximum. Further, it is assumed that the crack is initiated when $\overline{\sigma_{\theta\theta}}(\theta_0)$ reaches its critical value $\overline{\sigma_{\theta\theta\,C}}(\theta_0)$ that is ascertained for a crack in homogeneous media. The distance d has to be chosen in dependence on the mechanism of a rupture, e.g. as a dimension of a plastic zone or as a size of material grain.

Crack initiation direction

The potential direction of crack initiation is gained from the maximum of the mean value of the tangential stress in both materials. The following two conditions have to be satisfied:

$$\left(\frac{\partial\overline{\sigma_{m\theta\theta}}}{\partial\theta}\right)_{\theta_0} = 0 \qquad \left(\frac{\partial^2\overline{\sigma_{m\theta\theta}}}{\partial\theta^2}\right)_{\theta_0} < 0 \ , \tag{3}$$

Using (1) and (2) the average tangential stress for a bi-material wedge can be expressed as:

$$\overline{\sigma_{\theta\theta m}} = \frac{H_1}{\sqrt{2\pi}}\frac{d^{-p_1}}{(1-p_1)}F_{\theta\theta 1m} + \frac{H_2}{\sqrt{2\pi}}\frac{d^{-p_2}}{(1-p_2)}F_{\theta\theta 2m} \ , \tag{4}$$

and for its first derivation it follows:

$$\frac{d^{-p_1}}{1-p_1}\frac{\partial F_{\theta\theta 1m}}{\partial\theta} + \frac{H_2}{H_1}\frac{d^{-p_2}}{1-p_2}\frac{\partial F_{\theta\theta 2m}}{\partial\theta} = 0 \tag{5}$$

where for k = 1, 2:

$$\frac{\partial F_{\theta\theta km}}{\partial\theta} = (p_k{}^2 - 3p_k + 2)[(2-p_k)(a_{mk}\cos((2-p_k)\theta) +$$
$$-b_{mk}\sin((2-p_k)\theta) - p_k(c_{mk}\cos(p_k\theta) + d_{mk}\sin(p_k\theta))]$$

It is obvious that by inserting the ratio of H_2/H_1 (obtained from the numerical solution) into (5), we obtain a simple equation for the value of θ_0. The maximum of $\sigma_{m\theta\theta}$ can exist in both the material 1 in the interval (0; ω_1) and the material 2 in the interval ($-\omega_2$; 0). If there are more than one direction of possible crack initiation, it is necessary to consider all of them.

Stability criterion suggestion

Here we present the suggestion of the stability criterion based on the average stress calculated across a distance d from the wedge tip. The value of the average stress $\overline{\sigma_{\theta\theta}}$ corresponding to the bi-material wedge is calculated for the direction of θ_0 and it is compared with the critical stress $\overline{\sigma_{\theta\theta\,C}}$ corresponding to the crack [6].

For a crack in homogeneous material under mode I of loading we obtain (the direction of the assumed crack propagation θ_0 = 0):

$$\overline{\sigma_{\theta\theta\,C}} = \frac{2K_{I\mathrm{Crit}}}{\sqrt{2\pi d}} \ . \tag{6}$$

Note that $K_{I\mathrm{Crit}}$ is a material characteristic represented by the fracture toughness K_{IC} (in the case of brittle fracture) or by the threshold value K_{Ith} (for the fatigue failure initiation). To find the relation between $H_k(\sigma_{\mathrm{appl}})$ and $H_{k\mathrm{Crit}}(M_m)$, let us consider the fact that the ratio of the values H_1 and H_2 is constant for a given bi-material configuration and boundary conditions and does not depend on the value of the applied stress σ_{appl} and it holds

$$\Gamma_{21} = \frac{H_2}{H_1} = \frac{H_{2\mathrm{Crit}}}{H_{1\mathrm{Crit}}} \ , \left[\mathrm{m}^{p_2-p_1}\right].$$

This assumption is justified because when changing the value of the applied stress, only the absolute values of GSIFs change, but their ratio is constant even for the critical values $H_{2\mathrm{Crit}}$ /$H_{1\mathrm{Crit}}$. The ratio has no physical meaning, it just represents the contribution of particular singular terms to the stress distribution. Inserting the ratio Γ_{21} and the critical value H_{1C} into the relation (4) we get the critical value of the average tangential stress for a bi-material wedge. Following the assumption of the same mechanism of a rupture in both cases (crack and notch) we can compare it with the relations for a crack (6) and obtain an expression for $H_{1\mathrm{Crit}}$ value:

$$H_{1\mathrm{Crit}} = \frac{2K_{I\mathrm{Crit}}}{\dfrac{d^{\frac{1}{2}-p_1}}{1-p_1} F_{\theta\theta 1m}(\theta_0) + \Gamma_{21} \dfrac{d^{\frac{1}{2}-p_2}}{1-p_2} F_{\theta\theta 2m}(\theta_0)} . \qquad (7)$$

It is evident that the critical value $H_{1\mathrm{Crit}}$ depends on the value of $K_{I\mathrm{Crit}}$ represented by the fracture toughness K_{IC} or the threshold value $K_{I\mathrm{th}}$ which are material characteristics. The critical values are compared for the directions θ_0 of the assumed crack initiation that is in the case of a bi-material notch ascertained from the maximal tangential stress criterion. Because of that fact, the normal loading mode is predominant and thus the comparison with crack propagation characterized by K_{IC} or $K_{I\mathrm{th}}$ (both for mode *I*) is justified. The integrating distance *d* has to be chosen according to the mechanism of the rupture, e.g. for a cleavage fracture it can be set between 2 ÷ 5× grain size of the material.

Then the stability criterion can be suggested in the following form:

$$H_1(\sigma_{\mathrm{appl}}) < H_{1\mathrm{Crit}}(K_{I\mathrm{Crit}}) . \qquad (8)$$

The crack is not initiated at the tip of a bi-material notch if the value $H_1$of GSIF is lower than its critical value $H_{1\mathrm{Crit}}$. The value H_1 is determined from a numerical solution of a given bi-material configuration with given boundary conditions, geometry, and material properties of the bi-material wedge. The critical value $H_{1\mathrm{Crit}}$ is given by the relation (7).

Because the units of GSIFs H_1 and $H_{1\mathrm{Crit}}$ [MPa.m^{p1}] depend on geometry and material combination, it is advantageous to interpret the results of the stability criterion in terms of critical applied stress formulated as:

$$\sigma_{\mathrm{Crit}} = \sigma_{appl} \frac{H_{1\mathrm{Crit}}}{H_1(\sigma_{appl})} . \qquad (9)$$

Where σ_{appl} in the relation (9) is the stress applied in the numerical solution for the value H_1. No crack will be initiated at the bi-material wedge tip if the applied stress is lower than the critical stress:

$$\sigma_{appl} < \sigma_{\mathrm{Crit}} . \qquad (10)$$

III. APPLICATION

In real engineering constructions composed of compound solids or composite materials, failures usually start at the junctions of the material components. Especially in the case of fatigue loading of the structure, a crack is mostly initiated at the free surface.

The presented approach assessing the stability of the bi-material notch model can be well utilised for a number of applications in composite materials. Some of the construction points suitable to be modelled as the bi-material notch are shown in table I together with notes for bi-material notch geometrical configuration.

IV. DISCUSSION AND CONCLUSIONS

The model of a bi-material notch is suitable to simulate a number of construction points from which a failure is initiated at the junction of two materials at the free surface. They are layered composites, some configurations of fibres at the free surface, edges of protective layers etc. In the locations where the layers or fibres touch the surface of the composite body the singular stress concentrations occur. Such places are often responsible for crack initiation and consequently for the final failure of the whole body. Neither ignoring such places nor considering them to be cracks is correct. Ignoring the concentrators at the surface can have fatal consequences On the other hand, the power of the singularity generally differs from ½ (that holds for a crack) so the procedures of classic fracture mechanics cannot be used. The generalization of the standard procedures is outlined above. The advantage of the method is that only the common material characteristics (Young`s modulus, Poisson`s ratio, fracture toughness, threshold value of stress intensity factor for loading mode *I*) of particular components are put into the process of assessing the bi-material notch. However, when taking the material characteristics one must be careful because of a possible difference between the properties of a thin fibre and of the same material in a compact form.

ACKNOWLEDGMENT

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Table I. Construction points and the corresponding bi-material notch model configurations

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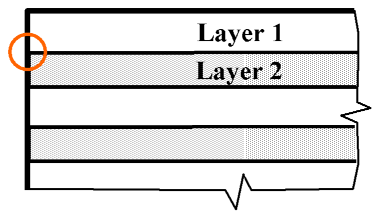
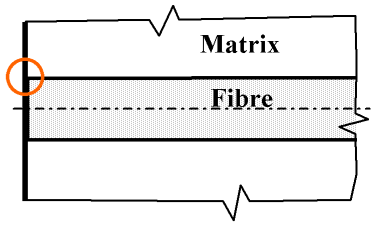
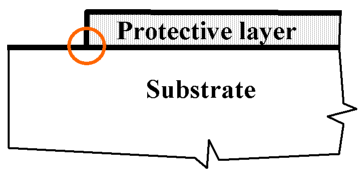
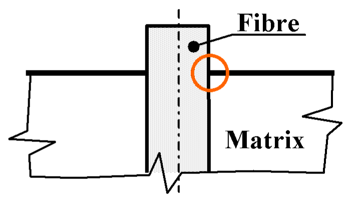
<tr><td>The stress concentrator</td><td></td><td></td></tr>
<tr><td>Geometrical configuration of bi-material notch</td><td>$\omega_1 = \omega_2 = 90°$
Plain strain</td><td>$\omega_1 = \omega_2 = 90°$
Rotate symmetry</td></tr>
<tr><td>The stress concentrator</td><td></td><td></td></tr>
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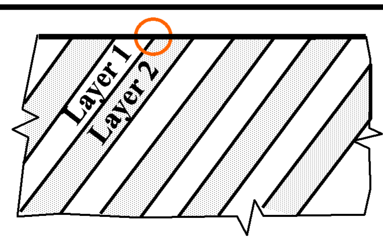
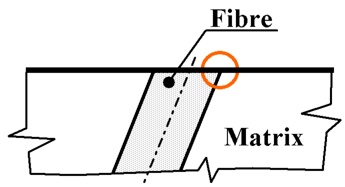
Plain strain</td><td>$\omega_1 = 90°$, $\omega_2 = 180°$
Rotate symmetry</td></tr>
<tr><td>The stress concentrator</td><td></td><td></td></tr>
<tr><td>Geometrical configuration of bi-material notch</td><td>$\omega_1 \neq 90°$, $\omega_2 = 180° - \omega_1$
Plain strain</td><td>$\omega_1 \neq 90°$, $\omega_2 = 180° - \omega_1$
Plain strain</td></tr>
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