Laminate Tube Stiffness Maximization by Winding Angle Control

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Problem of stiffness maximization

Let us consider a problem to maximize a stiffness measure of some construction by manipulating with material mechanics properties (i. e. choosing and changing a stiffness tensor). We search the stiffness tensor from a set of accessible stiffness tensors that maximize a stiffness measure.

There is a first question at this place. What is the stiffness measure? It is great philosophy issue, which is beyond the scope of this study, and therefore we select the following one:[1]

$$\mathfrak{s}(\boldsymbol{u}) = \frac{1}{\mathfrak{l}(\boldsymbol{u})}$$

where

$$\mathfrak{l}(\boldsymbol{u}) = \int_{\Omega} p_i u_i \,\mathrm{d}\Omega + \int_{\partial\Omega_t} t_i u_i \,\mathrm{d}S,$$

is potencial energy of external loads and

$$\boldsymbol{p}(\boldsymbol{x}) = \begin{pmatrix} p_x(\boldsymbol{x}) \\ p_y(\boldsymbol{x}) \\ p_z(\boldsymbol{x}) \end{pmatrix}$$

are mass forces that act in $\Omega \subset \mathbb{E}_3$, $\boldsymbol{x} = (x, y, z) = (x_1, x_2, x_3) \in \Omega$,

$$\boldsymbol{t}(\mathbf{x}) = \begin{pmatrix} t_x(\boldsymbol{x}) \\ t_y(\boldsymbol{x}) \\ t_z(\boldsymbol{x}) \end{pmatrix}$$

[1] See [BENDSØE, 1995] p. 7., [MAREŠ and HOLÝ, 2002], [MAREŠ, 2002], [MAREŠ, 2003a], [MAREŠ, 2003b].

are surface forces acting on boundary part $\partial\Omega_t \subset \partial\Omega$. Displacements are denoted by

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} u_x(\boldsymbol{x}) \\ u_y(\boldsymbol{x}) \\ u_z(\boldsymbol{x}) \end{pmatrix}.$$

Principle of minimum potential energy states that[2] *among all the admissible displacements which satisfy the prescibed geometrical boundary conditions, the actual displacements minimize the total potential energy:*

$$\Pi(\boldsymbol{u}) = \mathfrak{a}(\boldsymbol{u},\boldsymbol{u}) - \mathfrak{l}(\boldsymbol{u}), \tag{1}$$

where

$$\mathfrak{a}(\boldsymbol{u}) = \mathfrak{a}(\boldsymbol{u},\boldsymbol{u}) = \frac{1}{2}\int_\Omega E_{ijkl}(\boldsymbol{x}) \cdot \varepsilon_{ij}(\boldsymbol{u}(\boldsymbol{x})) \cdot \varepsilon_{kl}(\boldsymbol{u}(\boldsymbol{x}))\, \mathrm{d}\Omega$$

is elastic potential energy,

$$\sigma_{ij}(\boldsymbol{u}(\boldsymbol{x})) = E_{ijkl}(\boldsymbol{x}) \cdot \varepsilon_{kl}(\boldsymbol{u}(\boldsymbol{x}))$$

is generalized Hooke's Law and

$$\varepsilon_{ij}(\boldsymbol{u}(\boldsymbol{x})) = \frac{1}{2}\left(u_{i,j}(\boldsymbol{x}) + u_{j,i}(\boldsymbol{x})\right)$$

is Cauchy's tensor for small displacements.

For the actual displacements $\hat{\boldsymbol{u}}$ holds

$$\hat{\boldsymbol{u}} = \arg\min_{\boldsymbol{u}\in\mathbb{U}} \Pi(\boldsymbol{u}),$$

where $\mathbb{U}$ is the set of all the admissible displacements which satisfy the prescibed geometrical boundary conditions.

It is also

$$\Pi(\hat{\boldsymbol{u}}) = \mathfrak{a}(\hat{\boldsymbol{u}},\hat{\boldsymbol{u}}) - \mathfrak{l}(\hat{\boldsymbol{u}}) = \min_{\boldsymbol{u}\in\mathbb{U}} \Pi(\boldsymbol{u}) = \min_{\boldsymbol{u}\in\mathbb{U}}(\mathfrak{a}(\boldsymbol{u},\boldsymbol{u}) - \mathfrak{l}(\boldsymbol{u})).$$

From the theory of variational methods[3] it is known that in a equilibrium state

$$\min_u \Pi = \frac{1}{2}(A\hat{u},\hat{u}) - (f,\hat{u}) \;\wedge\; (A\hat{u},\hat{u}) = (f,\hat{u}) \quad\Rightarrow\quad \Pi(\hat{u}) = \min_u \Pi = -\frac{1}{2}(f,\hat{u}) = -\frac{1}{2}\mathfrak{l}(\hat{u})$$

and hence

$$\Pi(\hat{\boldsymbol{u}}) = -\frac{1}{2}\mathfrak{l}(\hat{\boldsymbol{u}}) < 0.$$

Since for $\hat{\boldsymbol{E}}$ that both maximize stiffness measure and minimize compliance measure

$$\hat{\boldsymbol{E}} = \arg\min_{\boldsymbol{E}} \mathfrak{l}(\hat{\boldsymbol{u}}) = \arg\max_{\boldsymbol{E}} \left(-\frac{1}{2}\mathfrak{l}(\hat{\boldsymbol{u}})\right) =$$

$$= \arg\max_{\boldsymbol{E}}(\mathfrak{a}(\hat{\boldsymbol{u}},\hat{\boldsymbol{u}}) - \mathfrak{l}(\hat{\boldsymbol{u}})) = \arg\max_{\boldsymbol{E}}\min_{\boldsymbol{u}}(\mathfrak{a}(\boldsymbol{u},\boldsymbol{u}) - \mathfrak{l}(\boldsymbol{u}))$$

is valid, we may, if we search stiffness tensor $\hat{\boldsymbol{E}} = \{\hat{E}_{ijkl}(\boldsymbol{x})\}$ that minimize compliance measure $\mathfrak{l}(\boldsymbol{u})$, solve the problem

$$\{\hat{\boldsymbol{E}},\hat{\boldsymbol{u}}\} = \arg\max_{\boldsymbol{E}\in\mathbb{E}}\min_{\boldsymbol{u}\in\mathbb{U}}(\mathfrak{a}(\boldsymbol{u},\boldsymbol{u}) - \mathfrak{l}(\boldsymbol{u})), \tag{2}$$

where $\mathbb{E}$ is set of acceptable stiffness tensors and $\mathbb{U}$ is set of all the admissible displacements which satisfy the prescibed geometrical boundary conditions.

[2] [Washizu, 1975] p. 27.

[3] See for e.g. [Вацудзу, 1987] and [Bendsøe, 1995].

At topology design the set $\mathbb{E}$ contains stiffness tensor of given isotropy material E_{ijkl} and *null* material, i.e., stiffness tensor with zero components. It is

$$E_{ijkl} = \delta(\boldsymbol{x}) E^0_{ijkl},$$

where

$$\delta(\boldsymbol{x}) = \begin{cases} 1 & \text{at the point } \boldsymbol{x} \text{ with material,} \\ 0 & \text{at the point } \boldsymbol{x} \text{ without material.} \end{cases}$$

Volume (weight) constraint is given by

$$\int_\Omega \delta(\boldsymbol{x}) \, \mathrm{d}\Omega \leq V_0.$$

In this case the indicate function $\delta(\boldsymbol{x})$ is the only one design variable.

In the case of linear elastic material it is posible to write the last problem by using the complementary energy as

$$\{\hat{\boldsymbol{C}}, \hat{\boldsymbol{\sigma}}\} = \arg \min_{\boldsymbol{C}\in\mathbb{C}} \min_{\boldsymbol{\sigma}\in\mathbb{S}} \left(\frac{1}{2} \int_\Omega C_{ijkl} \sigma_{ij} \sigma_{kl} \, \mathrm{d}\Omega \right), \tag{3}$$

where the set of all statically admissible stresses

$$\mathbb{S} = \{ \sigma_{ij} \mid \sigma_{ij,i} + p_j = 0 \text{ na } \Omega \quad \wedge \quad \sigma_{ij} \cdot \ell_j = t_i \text{ na } \partial\Omega_t \},$$

at this $\ell_j(\boldsymbol{x})$ is directional cosinus of outward normal to boundary $\partial\Omega$ of domain Ω at $\boldsymbol{x} \in \partial\Omega_t$ and $\boldsymbol{C} = \{C_{ijkl}\}$ is compliance tensor from where the set of all admissible compliance tensors $\mathbb{C}$. It holds that

$$\varepsilon_{ij} = C_{ijkl} \sigma_{kl}.$$

Constitutive law of thin laminate ply

Generalized Hooke's Law for laminate ply

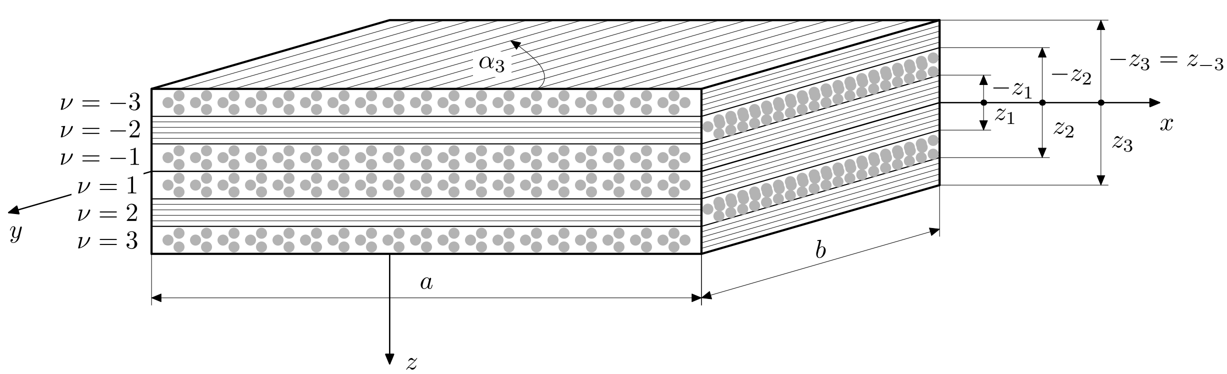


Fig. 2: Laminate plate composed of $2N = 2 \cdot 3$ orthotropic symetrically laid plies

Constitutive law of thin laminate ply in the main material coordinate system x^ν–y^ν of νth orthotropic ply is given by[4]

$$\begin{pmatrix} \sigma^\nu_{11} \\ \sigma^\nu_{22} \\ \sigma^\nu_{12} \end{pmatrix} = \begin{pmatrix} Q^\nu_{11} & Q^\nu_{12} & 0 \\ Q^\nu_{12} & Q^\nu_{22} & 0 \\ 0 & 0 & 2G^\nu_{12} \end{pmatrix} \begin{pmatrix} \varepsilon^\nu_{11} \\ \varepsilon^\nu_{22} \\ \varepsilon^\nu_{12} \end{pmatrix}, \tag{4}$$

[4] [GÜRDAL *et al.*, 1999] pp. 53, 63.

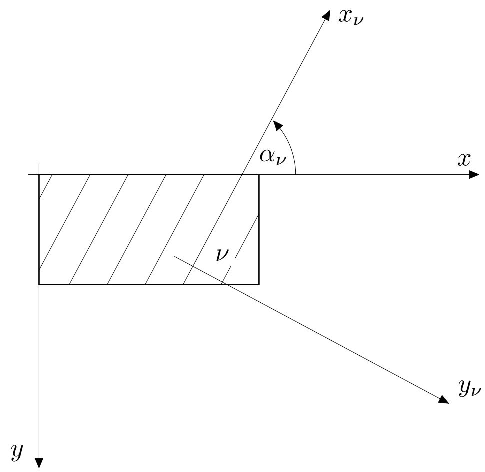


Fig. 3: Global coordinate system x–y and main material coordinate system x^ν–y^ν

where

$$Q_{11}^\nu = \frac{E_1^\nu}{1-\nu_{12}^\nu \nu_{21}^\nu}, \qquad \nu_{21}^\nu = \nu_{12}^\nu \frac{E_2^\nu}{E_1^\nu}, \tag{5}$$

$$Q_{22}^\nu = \frac{E_2^\nu}{1-\nu_{12}^\nu \nu_{21}^\nu}, \qquad Q_{12}^\nu = \frac{\nu_{12}^\nu E_2}{1-\nu_{12}^\nu \nu_{21}^\nu}, \tag{6}$$

and where $(\nu = 1, 2, \ldots, N)$ is the sign of ply sequence from the center of plate, and E_1^ν, E_2^ν a G_{12}^ν are moduli of elasticity in the major direction, in the minor direction of the νth ply, and modulus of elasticity in shear of the νth ply, respectively. For major Poisson ratio ν_{12} in the case of loading in only major direction x^ν it holds

$$\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1}.$$

Likewise for minor Poisson ratio in the case of loading in only minor direction y^ν

$$\nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2}.$$

It also holds

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}.$$

Constitutive law in two-dimensinal tensor notation

We consider the plane stress state for the formulation of constitutive law. Let us introduce stress tensor and strain tensor of νth ply at main material coordinate system of νth orthotropic ply x^ν–y^ν by[5]

$$\{\sigma_{ij}^\nu\} = \begin{pmatrix} \sigma_{11}^\nu & \sigma_{12}^\nu \\ \sigma_{21}^\nu & \sigma_{22}^\nu \end{pmatrix} = \begin{pmatrix} \sigma_{xx}^\nu & \sigma_{xy}^\nu \\ \sigma_{yx}^\nu & \sigma_{yy}^\nu \end{pmatrix},$$

$$\{\varepsilon_{ij}^\nu\} = \begin{pmatrix} \varepsilon_{11}^\nu & \varepsilon_{12}^\nu \\ \varepsilon_{21}^\nu & \varepsilon_{22}^\nu \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx}^\nu & \varepsilon_{xy}^\nu \\ \varepsilon_{yx}^\nu & \varepsilon_{yy}^\nu \end{pmatrix},$$

where alternatively, as in the future, we use equivalent indexing 11 or xx. Relation of stress–strain at main material coordinate system of νth orthotropic ply have form

$$\sigma_{ij}^\nu = E_{ijkl}^\nu \varepsilon_{kl}^\nu,$$

[5] See [Mareš, 2002], [Mareš, 2003a], [Mareš, 2003b], [Mareš, 2004].

where

$$\{E^{\nu}_{ijkl}\}_{ij\lceil kl} = \begin{pmatrix} Q^{\nu}_{11} & 0 & 0 & Q^{\nu}_{12} \\ 0 & G^{\nu}_{12} & G^{\nu}_{12} & 0 \\ 0 & G^{\nu}_{12} & G^{\nu}_{12} & 0 \\ Q^{\nu}_{12} & 0 & 0 & Q^{\nu}_{22} \end{pmatrix}, \tag{7}$$

and right-low index $ij\lceil kl$ says that rows of the matrix are related to $(ij = 11, 12, 21, 22)$ and columns to $(kl = 11, 12, 21, 22)$. This stress–strain relation is evident from following expanding of relations (4):

$$\sigma^{\nu}_{11} = Q^{\nu}_{11}\varepsilon^{\nu}_{11} + Q^{\nu}_{12}\varepsilon^{\nu}_{22}$$

$$\sigma^{\nu}_{22} = Q^{\nu}_{12}\varepsilon^{\nu}_{11} + Q^{\nu}_{22}\varepsilon^{\nu}_{22}$$

$$\sigma^{\nu}_{12} = \sigma^{\nu}_{21} = G^{\nu}_{12}\varepsilon^{\nu}_{12} + G^{\nu}_{12}\varepsilon^{\nu}_{21} = 2G^{\nu}_{12}\varepsilon^{\nu}_{12} = 2G^{\nu}_{12}\varepsilon^{\nu}_{21} \quad (\varepsilon_{12} = \varepsilon_{21}).$$

By tranforming these expression of stress and strain tensors from main material coordinate system of νth orthotropic ply x^{ν}–y^{ν} into global coordinate system of laminate plate x–y we state the stress tensor of νth ply in the global coordinate system x–y

$$\sigma_{ij}(x, y, z) = \ell^{\nu}_{ik}\ell^{\nu}_{jl}\sigma^{\nu}_{kl} \qquad \begin{array}{l} \forall\ z \in \langle z_{\nu-1}, z_{\nu}\rangle \text{ if } \nu > 0, \\ \forall\ z \in \langle z_{\nu}, z_{\nu+1}\rangle \text{ if } \nu < 0, \end{array}$$

as function of strain tensor of νth ply in the global coordinate system x–y

$$\varepsilon_{ij}(x, y, z) = \ell^{\nu}_{ik}\ell^{\nu}_{jl}\varepsilon^{\nu}_{kl} \qquad \begin{array}{l} \forall\ z \in \langle z_{\nu-1}, z_{\nu}\rangle \text{ if } \nu > 0, \\ \forall\ z \in \langle z_{\nu}, z_{\nu+1}\rangle \text{ if } \nu < 0. \end{array} \tag{8}$$

Transformation matrix has (see Figure 3) the form

$$\{\ell^{\nu}_{ik}\}_{i\lceil k} = \{\ell_{ik}(\alpha_{\nu})\}_{i\lceil k} = \{\cos(x_i, x^{\nu}_k)\}_{i\lceil k} = \begin{pmatrix} \cos(x, x^{\nu}) & \cos(x, y^{\nu}) \\ \cos(y, x^{\nu}) & \cos(y, y^{\nu}) \end{pmatrix} = \begin{pmatrix} \cos\alpha_{\nu} & \sin\alpha_{\nu} \\ -\sin\alpha_{\nu} & \cos\alpha_{\nu} \end{pmatrix}. \tag{9}$$

Since

$$\ell^{\nu}_{ik}\ell^{\nu}_{jk} = \delta_{ij}$$

and

$$\ell^{\nu}_{ki}\ell^{\nu}_{kj} = \delta_{ij},$$

where Kronecker δ

$$\delta_{kl} = \begin{cases} 1 & \text{for} \quad l = k, \\ 0 & \text{for} \quad l \neq k, \end{cases}$$

we have, using production of both sides of equality (8) by expression $\ell^{\nu}_{im}\ell^{\nu}_{jn}$, inverse transformation relation

$$\varepsilon^{\nu(z)}_{mn}(x, y, z) = \ell^{\nu(z)}_{im}\ell^{\nu(z)}_{jn}\varepsilon_{ij}(x, y, z),$$

where tensor ℓ^{ν}_{im} belongs to νth ply with $\nu = \nu(z)$ as follows: $z \in \langle z_{\nu-1}, z_{\nu}\rangle$ if $z > 0$, and if $z < 0$ then $z \in \langle z_{\nu}, z_{\nu+1}\rangle$.

Using the equations

$$\sigma_{ij}(x, y, z) = \ell^{\nu(z)}_{im}\ell^{\nu(z)}_{jn}\sigma^{\nu(z)}_{mn} = \ell^{\nu(z)}_{im}\ell^{\nu(z)}_{jn}E^{\nu(z)}_{mnop}\varepsilon^{\nu(z)}_{op},$$

the searched relation between stress tensor of νth ply in the global coordinate system and strain tensor of νth ply in the global coordinate system takes the form of constitutive law of thin laminate ply:

$$\sigma_{ij}(x, y, z) = \ell^{\nu(z)}_{im}\ell^{\nu(z)}_{jn}E^{\nu(z)}_{mnop}\ell^{\nu(z)}_{ko}\ell^{\nu(z)}_{lp}\varepsilon_{kl}(x, y, z).$$

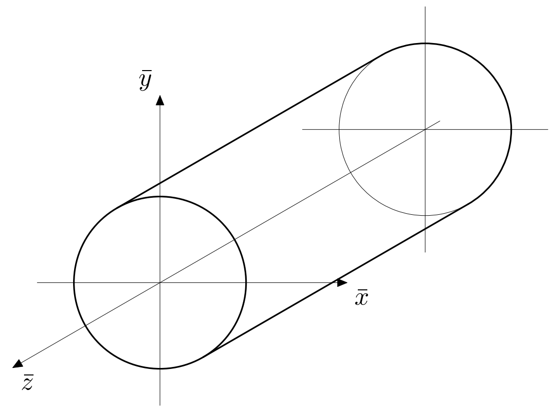


Fig. 4: Carthesian coordinate system $\bar{x}\bar{y}\bar{z}$ of the tube

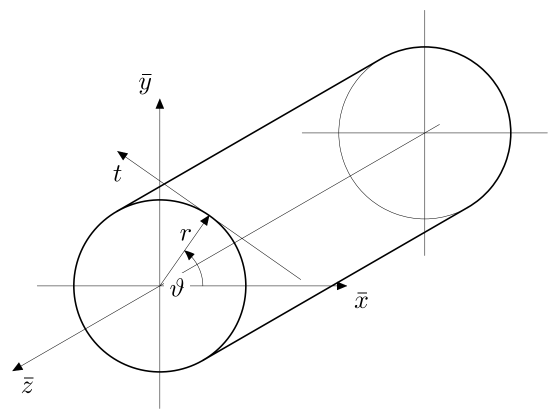


Fig. 5: Cylindrical coordinate system $rt\bar{z}$ of the tube

Optimization of laminate tube winding angle

Used coordinate systems

Transformation of two-dimensional stress tensor and strain tensor from the main material coordinate system into global coordinate system of the unrolled tube

At this subsection we put transformation of two-dimensional stress tensor σ_{ij}^{ν} and strain tensor ε_{mn}^{ν} in the main material coordinate system x^{ν}–y^{ν} into global coordinate system of the unrolled tube x–y (at which we denote them as σ_{ij}, ε_{kl}).

From above it is

$$\sigma_{ij} = \ell_{ik}^{\nu}\ell_{jl}^{\nu}\sigma_{kl}^{\nu},$$

where

$$\{\ell_{ik}^{\nu}\}_{i\lceil k} = \begin{pmatrix} \cos\alpha_{\nu} & \sin\alpha_{\nu} \\ -\sin\alpha_{\nu} & \cos\alpha_{\nu} \end{pmatrix}.$$

Once more from above

$$\varepsilon_{mn}^{\nu} = \ell_{im}^{\nu}\ell_{jn}^{\nu}\varepsilon_{ij}.$$

Thin tube (tube made from one orthotropic ply)

Premises

- Tube is made from one orthotropic ply ($\nu = 1$).

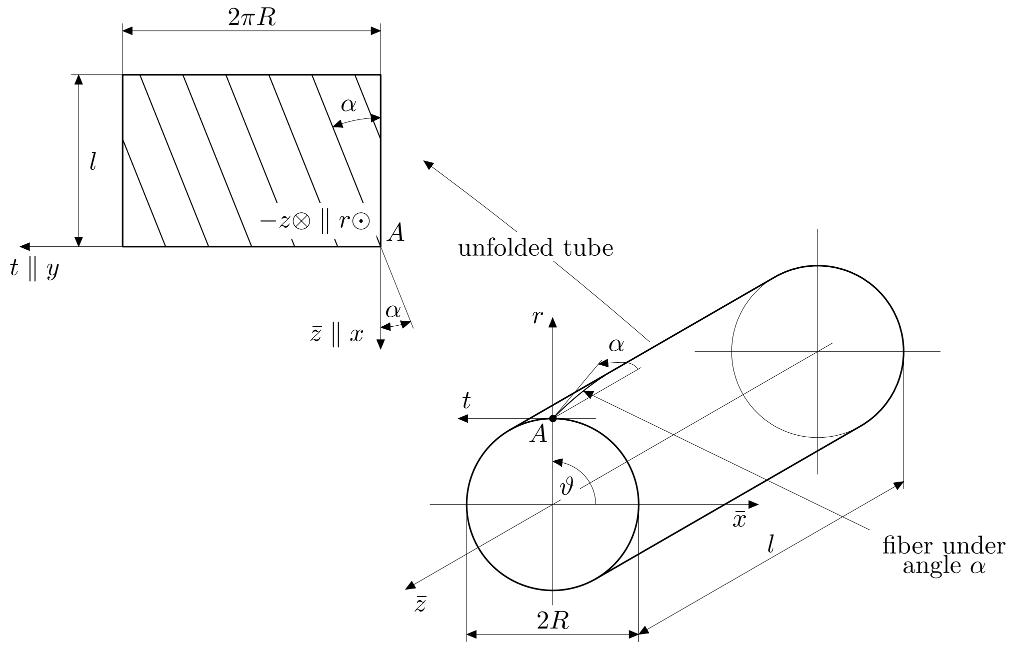


Fig. 6: Global coordinate system xyz of the unrolled tube

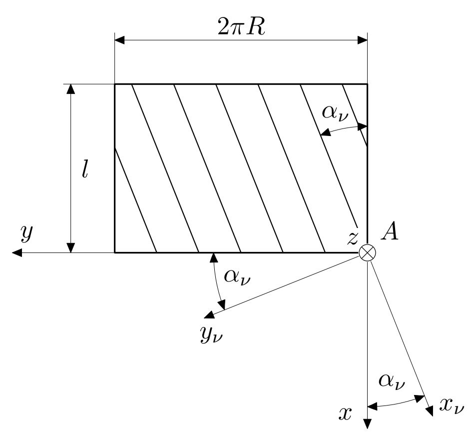


Fig. 7: Main material coordinate system $x_\nu y_\nu z_\nu$ of νth orthotropic ply

- Hooke's law of that ply in the main material coordinate system x^ν–y^ν takes the form $\sigma_{ij}^\nu = E_{ijkl}^\nu \varepsilon_{kl}^\nu$.
- We do not consider buckling.
- Ply thickness is negligible with respect to radius of tube: cylindrical coordinate system $rt\bar{z}$ of the tube coincide whit global coordinate system of the unrolled tube (xyz).

Torsion

We consider that buckling does not happen and that moment of torsion M_k is transmited by shear stress $\sigma_{\bar{z}t}$. Let this stress be (for the ply is thin-walled) constant throughout the cross-section, i.e.

$$M_k = 2\pi RT\sigma_{\bar{z}t}R.$$

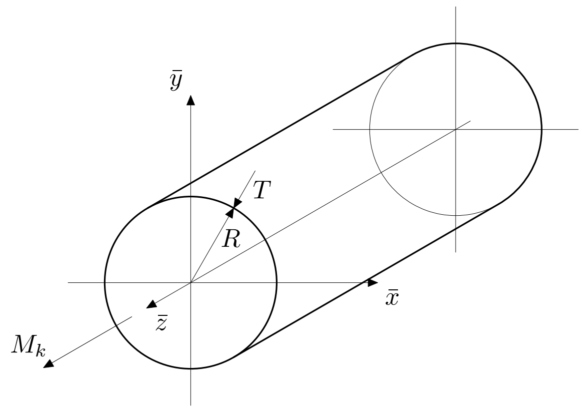


Fig. 8: One-ply laminate tube under torsion

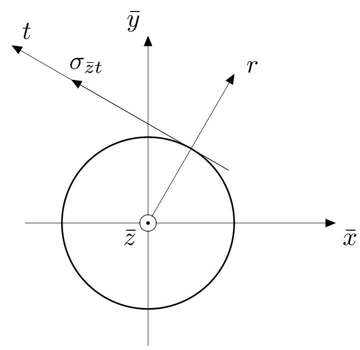


Fig. 9: Stress $\sigma_{\bar{z}t}$ in the one-ply laminate tube under torsion

At the global coordinate system of the unrolled tube (system xyz) the stress $\sigma_{\bar{z}t}$ is equivalent to component σ_{xy} of stress tensor. The others component are null. Hence

$$\{\sigma_{ij}\}_{i\lceil j} = \begin{pmatrix} 0 & \sigma_{xy} \\ \sigma_{xy} & 0 \end{pmatrix}.$$

Tension

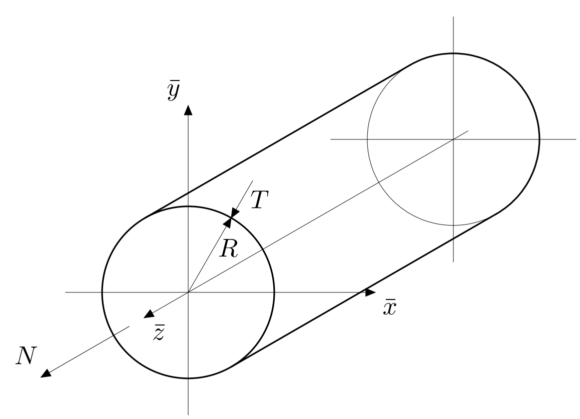


Fig. 10: One-ply laminate tube under tension

Let us consider

$$N = 2\pi R T \sigma_{\bar{z}\bar{z}},$$

for $\sigma_{\bar{z}\bar{z}}$ constant throughout the cross-section.

At the global coordinate system of the unrolled tube (system xyz) the stress $\sigma_{\bar{z}\bar{z}}$ is equivalent to component σ_{xx} of stress tensor. At the global coordinate system xyz the stress tensor takes (for simple tension) form

$$\{\sigma_{ij}\}_{i\lceil j} = \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{pmatrix}.$$

Bending

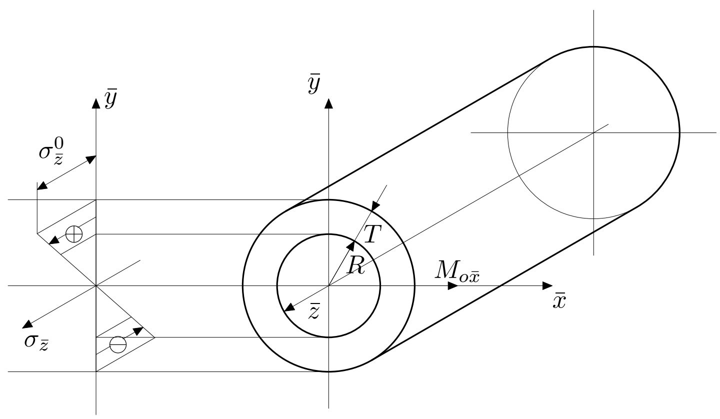


Fig. 11: One-ply laminate tube under bending

With the well-known assumption about linear distribution of stress under bending we have

$$M_{o\bar{x}} = \int_A \sigma^0_{\bar{z}} \cdot \frac{\bar{y}}{R} \cdot \bar{y} \, \mathrm{d}A = \frac{\sigma^0_{\bar{z}}}{R} J_{\bar{x}},$$

and thus

$$\sigma^0_{\bar{z}} = \frac{M_{o\bar{x}}}{J_{\bar{x}}} R$$

and

$$\sigma_{\bar{z}} = \frac{M_{o\bar{x}}}{J_{\bar{x}}} \bar{y}.$$

Also at this case it is $\sigma_{xx} = \sigma_{\bar{z}}$ and at the global coordinate system xyz

$$\{\sigma_{ij}\}_{i\lceil j} = \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{pmatrix}.$$

However we do not consider this case here since from the optimization point of view it is not diferent from the tension case.

Winding angle that maximize tube stiffness

Let us consider the above problem

$$\{\hat{\boldsymbol{C}}, \hat{\boldsymbol{\sigma}}\} = \arg \min_{\boldsymbol{C} \in \mathbb{C}} \min_{\boldsymbol{\sigma} \in \$} \frac{1}{2} \int_\Omega C_{ijkl} \sigma_{ij} \sigma_{kl} \, \mathrm{d}\Omega, \tag{10}$$

where

$$\$ = \{\sigma_{ij} \mid \sigma_{ij,i} + p_j = 0 \text{ na } \Omega, \quad \sigma_{ij} \ell_j = t_i \text{ na } \partial_t \Omega\}$$

and where $\mathbb{C}$ is a set of acceptable compliance tensors.

Compliance tensor of laminate ply ν

For the ply at the main material coordinate system x^ν–y^ν the following form holds

$$\varepsilon^\nu_{ij} = C^\nu_{ijkl}\sigma^\nu_{kl}.$$

From above we have

$$\varepsilon^\nu_{21} = \varepsilon^\nu_{12} = \frac{\sigma^\nu_{12}}{2G_{12}} = \frac{\sigma^\nu_{21}}{2G_{12}} \qquad (\sigma^\nu_{12} = \sigma^\nu_{21}),$$

$$\varepsilon^\nu_{21} = \varepsilon^\nu_{12} = \frac{1}{4G_{12}} \cdot \sigma^\nu_{12} + \frac{1}{4G_{12}} \cdot \sigma^\nu_{21}$$

and

$$\sigma^\nu_{11} = Q^\nu_{11}\varepsilon^\nu_{11} + Q^\nu_{12}\varepsilon^\nu_{22},$$
$$\sigma^\nu_{22} = Q^\nu_{12}\varepsilon^\nu_{11} + Q^\nu_{22}\varepsilon^\nu_{22},$$

from that

$$Q^\nu_{22}\sigma^\nu_{11} - Q^\nu_{12}\sigma^\nu_{22} = (Q^\nu_{11}Q^\nu_{22} - Q^\nu_{12}Q^\nu_{12})\,\varepsilon^\nu_{11},$$
$$Q^\nu_{11}\sigma^\nu_{22} - Q^\nu_{12}\sigma^\nu_{11} = (Q^\nu_{11}Q^\nu_{22} - Q^\nu_{12}Q^\nu_{12})\,\varepsilon^\nu_{22}$$

yielding

$$\begin{pmatrix} \varepsilon^\nu_{11} \\ \varepsilon^\nu_{12} \\ \varepsilon^\nu_{21} \\ \varepsilon^\nu_{22} \end{pmatrix} = \begin{pmatrix} C^\nu_{1111} & 0 & 0 & C^\nu_{1122} \\ 0 & C^\nu_{1212} & C^\nu_{1221} & 0 \\ 0 & C^\nu_{2112} & C^\nu_{2121} & 0 \\ C^\nu_{2211} & 0 & 0 & C^\nu_{2222} \end{pmatrix} \begin{pmatrix} \sigma^\nu_{11} \\ \sigma^\nu_{12} \\ \sigma^\nu_{21} \\ \sigma^\nu_{22} \end{pmatrix},$$

where

$$C^\nu_{1111} = \frac{Q^\nu_{22}}{Q^\nu_{11}Q^\nu_{22} - Q^\nu_{12}Q^\nu_{12}} = c^\nu_1,$$

$$C^\nu_{1122} = C^\nu_{2211} = \frac{Q^\nu_{12}}{Q^\nu_{12}Q^\nu_{12} - Q^\nu_{11}Q^\nu_{22}} = c^\nu_{12},$$

$$C^\nu_{2222} = \frac{Q^\nu_{11}}{Q^\nu_{11}Q^\nu_{22} - Q^\nu_{12}Q^\nu_{12}} = c^\nu_2,$$

$$C^\nu_{1212} = C^\nu_{1221} = C^\nu_{2112} = C^\nu_{2121} = \frac{1}{4G^\nu_{12}} = g^\nu.$$

Note: If

$$Q^\nu_{11}Q^\nu_{22} - Q^\nu_{12}Q^\nu_{12} < 0$$

then it is unusual material: Negative Poisson's ratio.[6]

For r respective z from ply ν we have (at the global coordinate system xyz)

$$\varepsilon_{ij}(x,y,z) = C_{ijkl}(z)\sigma_{kl}(x,y,z),$$

$$\varepsilon_{ij}(x,y,z) = \ell^{\nu(z)}_{ik}\ell^{\nu(z)}_{jl}\varepsilon^{\nu(z)}_{kl}(x,y,z) = \ell^{\nu(z)}_{ik}\ell^{\nu(z)}_{jl}C^{\nu(z)}_{klmn}\sigma^{\nu(z)}_{mn}(x,y,z),$$

$$\varepsilon_{ij}(x,y,z) = \ell^{\nu(z)}_{ik}\ell^{\nu(z)}_{jl}C^{\nu(z)}_{klmn}\ell^{\nu(z)}_{om}\ell^{\nu(z)}_{pn}\sigma_{op}(x,y,z),$$

or, at the form $\varepsilon_{ij}(x,y,z) = C_{ijkl}(z)\sigma_{kl}(x,y,z)$, as folowing

$$\varepsilon_{ij}(x,y,z) = \ell^{\nu(z)}_{im}\ell^{\nu(z)}_{jn}C^{\nu(z)}_{mnop}\ell^{\nu(z)}_{ko}\ell^{\nu(z)}_{lp}\sigma_{kl}(x,y,z).$$

From that we obtain

$$C_{ijkl}(z) = \ell^{\nu(z)}_{im}\ell^{\nu(z)}_{jn}C^{\nu(z)}_{mnop}\ell^{\nu(z)}_{ko}\ell^{\nu(z)}_{lp}.$$

[6] See for example [Friis *et al.*, 1988].

Single Ply Laminate Tube under Tension, Torsion and Interior Pressure

At this place we investigate problem

$$\left\{\hat{\boldsymbol{C}},\hat{\boldsymbol{\sigma}}\right\} = \arg\min_{\boldsymbol{C}\in\mathbb{C}}\min_{\boldsymbol{\sigma}\in\mathbb{S}}\frac{1}{2}\int_\Omega C_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}\Omega, \tag{11}$$

where $\mathbb{C}$ is the set of acceptable compliance tensors.

$$\mathbb{S} = \left\{\sigma_{xx} = \frac{N}{2\pi Rt},\ \sigma_{yy} = \frac{pR}{t},\ \sigma_{xy} = \frac{M_k}{2\pi R^2 t}\right\}.$$

For the compliance tensor C_{ijkl} at global coordinate system (for $r \in \langle R+(\nu-1)t, R+\nu t\rangle$ i.e. for νth ply) it holds that

$$C_{ijkl}\sigma_{ij}\sigma_{kl} = \ell^\nu_{im}\ell^\nu_{jn}\ell^\nu_{ko}\ell^\nu_{lp}C^\nu_{mnop}\sigma_{ij}\sigma_{kl}$$

with

$$\ell^\nu_{im} = \delta_{im}c_\nu + \epsilon_{im}s_\nu,$$

where

$$c_\nu = \cos\alpha_\nu, \quad s_\nu = \sin\alpha_\nu.$$

Consequently

$$\begin{aligned} C_{ijkl}\sigma_{ij}\sigma_{kl} = \big(&c^4_\nu\delta^\nu_{im}\delta^\nu_{jn}\delta^\nu_{ko}\delta^\nu_{lp} + c^3_\nu s_\nu\left(\epsilon^\nu_{im}\delta^\nu_{jn}\delta^\nu_{ko}\delta^\nu_{lp} + \delta^\nu_{im}\epsilon^\nu_{jn}\delta^\nu_{ko}\delta^\nu_{lp} + \delta^\nu_{im}\delta^\nu_{jn}\epsilon^\nu_{ko}\delta^\nu_{lp} + \delta^\nu_{im}\delta^\nu_{jn}\delta^\nu_{ko}\epsilon^\nu_{lp}\right) + \\ &+c^2_\nu s^2_\nu\left(\epsilon^\nu_{im}\epsilon^\nu_{jn}\delta^\nu_{ko}\delta^\nu_{lp} + \epsilon^\nu_{im}\delta^\nu_{jn}\epsilon^\nu_{ko}\delta^\nu_{lp} + \epsilon^\nu_{im}\delta^\nu_{jn}\delta^\nu_{ko}\epsilon^\nu_{lp} + \delta^\nu_{im}\epsilon^\nu_{jn}\epsilon^\nu_{ko}\delta^\nu_{lp} + \delta^\nu_{im}\epsilon^\nu_{jn}\delta^\nu_{ko}\epsilon^\nu_{lp} + \delta^\nu_{im}\delta^\nu_{jn}\epsilon^\nu_{ko}\epsilon^\nu_{lp}\right) + \\ &+c_\nu s^3_\nu\left(\delta^\nu_{im}\epsilon^\nu_{jn}\epsilon^\nu_{ko}\epsilon^\nu_{lp} + \epsilon^\nu_{im}\delta^\nu_{jn}\epsilon^\nu_{ko}\epsilon^\nu_{lp} + \epsilon^\nu_{im}\epsilon^\nu_{jn}\delta^\nu_{ko}\epsilon^\nu_{lp} + \epsilon^\nu_{im}\epsilon^\nu_{jn}\epsilon^\nu_{ko}\delta^\nu_{lp}\right) + s^4_\nu\epsilon^\nu_{im}\epsilon^\nu_{jn}\epsilon^\nu_{ko}\epsilon^\nu_{lp}\big)\,C^\nu_{mnop}\sigma_{ij}\sigma_{kl}. \end{aligned}$$

Let us write

$$C_{ijkl}\sigma_{ij}\sigma_{kl} = \mathfrak{R}^{1\nu}c^4_\nu + \mathfrak{R}^{2\nu}c^3_\nu s_\nu + \mathfrak{R}^{3\nu}c^2_\nu s^2_\nu + \mathfrak{R}^{4\nu}c^1_\nu s^3_\nu + \mathfrak{R}^{5\nu}s^4_\nu, \tag{12}$$

where coefficients $\mathfrak{R}^{\varrho\nu}$ $(\varrho = 1, 2, \dots, 5)$ are assembled as follows:

$$\mathfrak{R}^{1\nu} = C^\nu_{ijkl}\sigma_{ij}\sigma_{kl}.$$

$$\mathfrak{R}^{2\nu} = \left(\epsilon_{im}C^\nu_{mjkl} + \epsilon_{jn}C^\nu_{inkl} + \epsilon_{ko}C^\nu_{ijol} + \epsilon_{lp}C^\nu_{ijkp}\right)\sigma_{ij}\sigma_{kl}$$

and, since $C^\nu_{mnop} = C^\nu_{opmn} = C^\nu_{pomn}$ and $\sigma_{ij} = \sigma_{ji}$, it is

$$\mathfrak{R}^{2\nu} = 4\epsilon_{im}C^\nu_{mjkl}\sigma_{ij}\sigma_{kl}.$$

Coefficient of term $c^2_\nu s^2_\nu$:

$$\begin{aligned} \mathfrak{R}^{3\nu} = C^\nu_{mnop}\sigma_{ij}\sigma_{kl}\,(&(\delta_{ij}\delta_{mn} - \delta_{in}\delta_{jm})\,\delta_{ko}\delta_{lp} + (\delta_{ik}\delta_{mo} - \delta_{io}\delta_{mk})\,\delta_{jn}\delta_{lp} + (\delta_{il}\delta_{mp} - \delta_{ip}\delta_{ml})\,\delta_{jn}\delta_{ko} + \\ &+ (\delta_{jk}\delta_{no} - \delta_{jo}\delta_{nk})\,\delta_{im}\delta_{lp} + (\delta_{jn}\delta_{np} - \delta_{jp}\delta_{nl})\,\delta_{im}\delta_{ko} + (\delta_{kl}\delta_{op} - \delta_{kp}\delta_{ol})\,\delta_{im}\delta_{jn})\,, \end{aligned}$$

$$\begin{aligned} \mathfrak{R}^{3\nu} = {}&C^\nu_{mmkl}\sigma_{ii}\sigma_{kl} - C^\nu_{ijkl}\sigma_{ij}\sigma_{kl} + C^\nu_{mjml}\sigma_{kj}\sigma_{kl} - C^\nu_{kjil}\sigma_{ij}\sigma_{kl} + C^\nu_{mjkm}\sigma_{ij}\sigma_{ki} - C^\nu_{ljki}\sigma_{ij}\sigma_{kl} + C^\nu_{innl}\sigma_{ik}\sigma_{kl} - \\ &-C^\nu_{ikjl}\sigma_{ij}\sigma_{kl} + C^\nu_{ipkp}\sigma_{ij}\sigma_{kj} - C^\nu_{ilkj}\sigma_{ij}\sigma_{kl} + C^\nu_{ijpp}\sigma_{ij}\sigma_{kk} - C^\nu_{ijlk}\sigma_{ij}\sigma_{kl}, \end{aligned}$$

$$\mathfrak{R}^{3\nu} = 2C^\nu_{mmkl}\sigma_{ii}\sigma_{kl} - 2C^\nu_{ijkl}\sigma_{ij}\sigma_{kl} + 4C^\nu_{mjml}\sigma_{kj}\sigma_{kl} - 4C^\nu_{kjil}\sigma_{ij}\sigma_{kl},$$

and finally

$$\mathfrak{R}^{3\nu} = \left(2C^\nu_{mmkl}\delta_{ij} - 2C^\nu_{ijkl} + 4C^\nu_{mjml}\delta_{ik} - 4C^\nu_{kjil}\right)\sigma_{ij}\sigma_{kl}.$$

For $\mathfrak{R}^{4\nu}$ it holds

$$\mathfrak{R}^{4\nu} = C^\nu_{mnop}\sigma_{ij}\sigma_{kl}\left((\delta_{kl}\delta_{op} - \delta_{kp}\delta_{ol})\,\delta_{im}\epsilon_{jn} + (\delta_{kl}\delta_{op} - \delta_{kp}\delta_{ol})\,\epsilon_{im}\delta_{jn} + \right.$$

$$+\left(\delta_{jl}\delta_{np}-\delta_{jp}\delta_{nl}\right)\epsilon_{im}\delta_{ko}+\left(\delta_{jk}\delta_{no}-\delta_{jo}\delta_{nk}\right)\epsilon_{im}\delta_{lp}\right),$$

$$\mathfrak{R}^{4\nu}=\epsilon_{jn}C^{\nu}_{inpp}\sigma_{ij}\sigma_{kk}-\epsilon_{jn}C^{\nu}_{inkl}\sigma_{ij}\sigma_{kl}+\epsilon_{im}C^{\nu}_{mjpp}\sigma_{ij}\sigma_{kk}-\epsilon_{im}C^{\nu}_{mjkl}\sigma_{ij}\sigma_{kl}+\epsilon_{im}C^{\nu}_{mpkp}\sigma_{il}\sigma_{kl}-$$

$$-\epsilon_{im}C^{\nu}_{mlkj}\sigma_{ij}\sigma_{kl}+\epsilon_{im}C^{\nu}_{mool}\sigma_{ik}\sigma_{kl}-\epsilon_{im}C^{\nu}_{mkjl}\sigma_{ij}\sigma_{kl},$$

$$\mathfrak{R}^{4\nu}=\epsilon_{im}\left(2C^{\nu}_{jmpp}\sigma_{ij}\sigma_{kk}+2C^{\nu}_{mppk}\sigma_{il}\sigma_{kl}-2C^{\nu}_{jmkl}\sigma_{ij}\sigma_{kl}-2C^{\nu}_{mlkj}\sigma_{ij}\sigma_{kl}\right),$$

$$\mathfrak{R}^{4\nu}=\epsilon_{im}\left(2C^{\nu}_{jmpp}\delta_{kl}+2C^{\nu}_{mppk}\delta_{jl}-2C^{\nu}_{jmkl}-2C^{\nu}_{mlkj}\right)\sigma_{ij}\sigma_{kl}.$$

At last

$$\mathfrak{R}^{5\nu}=\epsilon_{im}\epsilon_{jn}\epsilon_{ko}\epsilon_{lp}C^{\nu}_{mnop}\sigma_{ij}\sigma_{kl},$$

$$\mathfrak{R}^{5\nu}=\left(\delta_{ij}\delta_{mn}-\delta_{in}\delta_{mj}\right)\left(\delta_{kl}\delta_{op}-\delta_{kp}\delta_{ol}\right)C^{\nu}_{mnop}\sigma_{ij}\sigma_{kl},$$

$$\mathfrak{R}^{5\nu}=C^{\nu}_{mmpp}\sigma_{ii}\sigma_{kk}+C^{\nu}_{ijkl}\sigma_{ij}\sigma_{kl}-C^{\nu}_{ijpp}\sigma_{ij}\sigma_{kk}-C^{\nu}_{mmkl}\sigma_{ii}\sigma_{kl},$$

$$\mathfrak{R}^{5\nu}=C^{\nu}_{mmpp}\sigma_{ii}\sigma_{kk}+C^{\nu}_{ijkl}\sigma_{ij}\sigma_{kl}-2C^{\nu}_{ijpp}\sigma_{ij}\sigma_{kk},$$

$$\mathfrak{R}^{5\nu}=\left(C^{\nu}_{mmpp}\delta_{ij}\delta_{kl}+C^{\nu}_{ijkl}-2C^{\nu}_{ijpp}\delta_{kl}\right)\sigma_{ij}\sigma_{kl}.$$

At the case of one-ply tube the set $\mathbb{S}$ has only one component (the solution $\hat{\boldsymbol{\sigma}}$ is known). Accordingly the coefficients $\mathfrak{R}^{\varrho\nu}$ are known numbers and the problem (11) has the form

$$\hat{\boldsymbol{\alpha}}=\arg\min_{\alpha_\nu\in\langle-\frac{\pi}{2},\frac{\pi}{2}\rangle}\frac{V}{2}\left(\mathfrak{R}^{1\nu}c_\nu^4+\mathfrak{R}^{2\nu}c_\nu^3s_\nu+\mathfrak{R}^{3\nu}c_\nu^2s_\nu^2+\mathfrak{R}^{4\nu}c_\nu s_\nu^3+\mathfrak{R}^{5\nu}s_\nu^4\right),$$

where V is the tube volume.

The necessary condition

$$\frac{\partial(\cdot)}{\partial\alpha_\nu}=0$$

has the form

$$\mathfrak{R}^{2\nu}c_\nu^4+\left(2\mathfrak{R}^{3\nu}-4\mathfrak{R}^{1\nu}\right)c_\nu^3s_\nu+\left(3\mathfrak{R}^{4\nu}-3\mathfrak{R}^{2\nu}\right)c_\nu^2s_\nu^2+\left(4\mathfrak{R}^{5\nu}-2\mathfrak{R}^{3\nu}\right)c_\nu s_\nu^3-\mathfrak{R}^{4\nu}s_\nu^4=0. \tag{13}$$

If $s_\nu=0$, it is $c_\nu=\pm1$ and $\mathfrak{R}^{2\nu}=0$. It is not truth. Hence it is not $s_\nu=0$. If $c_\nu=0$, it is $s_\nu=\pm1$ and $\mathfrak{R}^{4\nu}=0$. It also is not truth. Therefore it is not $c_\nu=0$. Since $c_\nu s_\nu\neq0$ we may divide equation (13) by c_ν^4, from that

$$\mathfrak{S}^{1\nu}\mathrm{tg}^4\alpha_\nu+\mathfrak{S}^{2\nu}\mathrm{tg}^3\alpha_\nu+\mathfrak{S}^{3\nu}\mathrm{tg}^2\alpha_\nu+\mathfrak{S}^{4\nu}\mathrm{tg}\,\alpha_\nu+\mathfrak{S}^{5\nu}=0, \tag{14}$$

where

$$\mathfrak{S}^{1\nu}=-\mathfrak{R}^{4\nu}=-\mathcal{R}^{4\nu}_{ijkl}\sigma_{ij}\sigma_{kl},$$

hereat

$$\mathcal{R}^{4\nu}_{ijkl}=2\epsilon_{im}\left(C^{\nu}_{jmpp}\delta_{kl}+C^{\nu}_{mppk}\delta_{jl}-C^{\nu}_{jmkl}-C^{\nu}_{mlkj}\right),$$

$$\left\{C^{\nu}_{mjkl}\right\}_{mj\lceil kl}=\begin{pmatrix}c_1^\nu&0&0&c_{12}^\nu\\0&g^\nu&g^\nu&0\\0&g^\nu&g^\nu&0\\c_{12}^\nu&0&0&c_2^\nu\end{pmatrix},$$

$$\left\{C^{\nu}_{mlkj}\right\}_{mj\lceil kl}=\begin{pmatrix}c_1^\nu&0&0&g^\nu\\0&g^\nu&c_{12}^\nu&0\\0&c_{12}^\nu&g^\nu&0\\g^\nu&0&0&c_2^\nu\end{pmatrix},$$

$$\left\{C^{\nu}_{mjpp}\right\}_{mj\lceil} = \begin{pmatrix} c^{\nu}_{1} + c^{\nu}_{12} \\ 0 \\ 0 \\ c^{\nu}_{12} + c^{\nu}_{2} \end{pmatrix},$$

$$\left\{C^{\nu}_{mppk}\right\}_{m\lceil k} = \begin{pmatrix} c^{\nu}_{1} + g^{\nu} & 0 \\ 0 & g^{\nu} + c^{\nu}_{2} \end{pmatrix},$$

and then

$$\left\{C^{\nu}_{jmpp}\delta_{kl} + C^{\nu}_{mppk}\delta_{jl} - C^{\nu}_{jmkl} - C^{\nu}_{mlkj}\right\}_{mj\lceil kl} = \begin{pmatrix} c^{\nu}_{12} + g^{\nu} & 0 & 0 & c^{\nu}_{1} - g^{\nu} \\ 0 & c^{\nu}_{1} - g^{\nu} & -c^{\nu}_{12} - g^{\nu} & 0 \\ 0 & -c^{\nu}_{12} - g^{\nu} & c^{\nu}_{2} - g^{\nu} & 0 \\ c^{\nu}_{2} - g^{\nu} & 0 & 0 & c^{\nu}_{12} + g^{\nu} \end{pmatrix},$$

$$\left\{\mathcal{R}^{4\nu}_{ijkl}\right\}_{ij\lceil kl} = \begin{pmatrix} 0 & -2c^{\nu}_{12} - 2g^{\nu} & 2c^{\nu}_{2} - 2g^{\nu} & 0 \\ 2c^{\nu}_{2} - 2g^{\nu} & 0 & 0 & 2c^{\nu}_{12} + 2g^{\nu} \\ -2c^{\nu}_{12} - 2g^{\nu} & 0 & 0 & -2c^{\nu}_{1} + 2g^{\nu} \\ 0 & -2c^{\nu}_{1} + 2g^{\nu} & 2c^{\nu}_{12} + 2g^{\nu} & 0 \end{pmatrix}.$$

Since

$$\left\{\sigma_{ij}\sigma_{kl}\right\}_{ij\lceil kl} = \begin{pmatrix} \sigma^{2}_{xx} & \sigma_{xx}\sigma_{xy} & \sigma_{xx}\sigma_{xy} & \sigma_{xx}\sigma_{yy} \\ \sigma_{xx}\sigma_{xy} & \sigma^{2}_{xy} & \sigma^{2}_{xy} & \sigma_{xy}\sigma_{yy} \\ \sigma_{xx}\sigma_{xy} & \sigma^{2}_{xy} & \sigma^{2}_{xy} & \sigma_{xy}\sigma_{yy} \\ \sigma_{yy}\sigma_{xx} & \sigma_{yy}\sigma_{xy} & \sigma_{yy}\sigma_{xy} & \sigma^{2}_{yy} \end{pmatrix}$$

finally it is

$$\mathfrak{S}^{1\nu} = 2\sigma_{xx}\sigma_{xy}\left(4g^{\nu} + 2c^{\nu}_{12} - 2c^{\nu}_{2}\right) + 2\sigma_{yy}\sigma_{xy}\left(-4g^{\nu} - 2c^{\nu}_{12} + 2c^{\nu}_{1}\right).$$

Furthermore we have

$$\mathfrak{S}^{2\nu} = 4\mathfrak{R}^{5\nu} - 2\mathfrak{R}^{3\nu}, \tag{15}$$

where

$$\mathfrak{R}^{5\nu} = \mathcal{R}^{5\nu}_{ijkl}\sigma_{ij}\sigma_{kl},$$

$$\mathcal{R}^{5\nu}_{ijkl} = \left(C^{\nu}_{mmpp}\delta_{ij}\delta_{kl} + C^{\nu}_{ijkl} - 2C^{\nu}_{ijpp}\delta_{kl}\right)$$

and

$$C^{\nu}_{mmpp} = c^{\nu}_{1} + 2c^{\nu}_{12} + c^{\nu}_{2},$$

$$\left\{C^{\nu}_{ijpp}\right\}_{ij\lceil} = \begin{pmatrix} c^{\nu}_{1} + c^{\nu}_{12} \\ 0 \\ 0 \\ c^{\nu}_{12} + c^{\nu}_{2} \end{pmatrix},$$

from which

$$\left\{\mathcal{R}^{5\nu}_{ijkl}\right\}_{ij\lceil kl} = \begin{pmatrix} c^{\nu}_{2} & 0 & 0 & -c^{\nu}_{1} + c^{\nu}_{12} + c^{\nu}_{2} \\ 0 & g^{\nu} & g^{\nu} & 0 \\ 0 & g^{\nu} & g^{\nu} & 0 \\ c^{\nu}_{1} + c^{\nu}_{12} - c^{\nu}_{2} & 0 & 0 & c^{\nu}_{1} \end{pmatrix}$$

and

$$\mathfrak{R}^{5\nu} = c^{\nu}_{2}\sigma^{2}_{xx} + 4g^{\nu}\sigma^{2}_{xy} + c^{\nu}_{1}\sigma^{2}_{yy} + 2c^{\nu}_{12}\sigma_{xx}\sigma_{yy}.$$

For coefficient $\mathfrak{R}^{3\nu}$ it is

$$\mathfrak{R}^{3\nu} = \mathcal{R}^{3\nu}_{ijkl}\sigma_{ij}\sigma_{kl},$$

where

$$\mathcal{R}^{3\nu}_{ijkl} = 2C^{\nu}_{mmkl}\delta_{ij} + 4C^{\nu}_{mjml}\delta_{ik} - 2C^{\nu}_{ijkl} - 4C^{\nu}_{kjil},$$

from which

$$\left\{\mathcal{R}^{3\nu}_{ijkl}\right\}_{ij\lceil kl} = \begin{pmatrix} 2c^{\nu}_{12}+4g^{\nu} & 0 & 0 & 2c^{\nu}_{2}-4g^{\nu} \\ 0 & 4c^{\nu}_{2}-2g^{\nu} & -4c^{\nu}_{12}-2g^{\nu} & 0 \\ 0 & -4c^{\nu}_{12}-2g^{\nu} & 4c^{\nu}_{1}-2g^{\nu} & 0 \\ 2c^{\nu}_{1}-4g^{\nu} & 0 & 0 & 2c^{\nu}_{12}+4g^{\nu} \end{pmatrix}$$

and hence

$$\mathfrak{R}^{3\nu} = \sigma^2_{xx}\left(2c^{\nu}_{12}+4g^{\nu}\right)+\sigma^2_{xy}\left(4c^{\nu}_{2}-8g^{\nu}-8c^{\nu}_{12}+4c^{\nu}_{1}\right)+\sigma^2_{yy}\left(2c^{\nu}_{12}+4g^{\nu}\right)+\sigma_{xx}\sigma_{yy}\left(2c^{\nu}_{2}-8g^{\nu}+2c^{\nu}_{1}\right).$$

Finally from (15) and from above

$$\mathfrak{S}_{2\nu} = \sigma^2_{xx}\left(4c^{\nu}_{2}-4c^{\nu}_{12}-8g^{\nu}\right)+\sigma^2_{xy}\left(32g^{\nu}-8c^{\nu}_{2}+16c^{\nu}_{12}-8c^{\nu}_{1}\right)+$$
$$+\sigma^2_{yy}\left(4c^{\nu}_{1}-4c^{\nu}_{12}-8g^{\nu}\right)+\sigma_{xx}\sigma_{yy}\left(16g^{\nu}-4c^{\nu}_{2}+8c^{\nu}_{12}-4c^{\nu}_{1}\right).$$

At the case of $\mathfrak{S}^{3\nu}$ we have

$$\mathfrak{S}^{3\nu} = 3\mathfrak{R}^{4\nu}-3\mathfrak{R}^{2\nu}.$$

From above

$$\mathfrak{R}^{4\nu} = 4\sigma_{xx}\sigma_{xy}\left(c^{\nu}_{2}-c^{\nu}_{12}-2g^{\nu}\right)+4\sigma_{yy}\sigma_{xy}\left(-c^{\nu}_{1}+c^{\nu}_{12}+2g^{\nu}\right)$$
$$\mathfrak{R}^{2\nu} = \mathcal{R}^{2\nu}_{ijkl}\sigma_{ij}\sigma_{kl},$$
$$\mathcal{R}^{2\nu}_{ijkl} = 4\epsilon_{im}C^{\nu}_{mjkl},$$

and

$$\left\{\mathcal{R}^{2\nu}_{ijkl}\right\}_{ij\lceil kl} = \begin{pmatrix} 0 & 4g^{\nu} & 4g^{\nu} & 0 \\ 4c^{\nu}_{12} & 0 & 0 & 4c^{\nu}_{2} \\ -4c^{\nu}_{1} & 0 & 0 & -4c^{\nu}_{12} \\ 0 & -4g^{\nu} & -4g^{\nu} & 0 \end{pmatrix}.$$

Again

$$\mathfrak{R}^{2\nu} = \sigma_{xx}\sigma_{xy}\left(8g^{\nu}+4c^{\nu}_{12}-4c^{\nu}_{1}\right)+\sigma_{yy}\sigma_{xy}\left(-8g^{\nu}-4c^{\nu}_{12}+4c^{\nu}_{2}\right)$$

and hence

$$\mathfrak{S}^{3\nu} = 12\sigma_{xx}\sigma_{xy}\left(c^{\nu}_{2}-2c^{\nu}_{12}-4g^{\nu}+c^{\nu}_{1}\right)+12\sigma_{yy}\sigma_{xy}\left(-c^{\nu}_{2}+2c^{\nu}_{12}+4g^{\nu}-c^{\nu}_{1}\right).$$

For fourth coefficient of condition (14) it is

$$\mathfrak{S}^{4\nu} = \left(2\mathfrak{R}^{3\nu}-4\mathfrak{R}^{1\nu}\right),$$

where

$$\mathfrak{R}^{3\nu} = \sigma^2_{xx}\left(2c^{\nu}_{12}+4g^{\nu}\right)+\sigma^2_{xy}\left(4c^{\nu}_{2}-8g^{\nu}-8c^{\nu}_{12}+4c^{\nu}_{1}\right)+\sigma^2_{yy}\left(2c^{\nu}_{12}+4g^{\nu}\right)+\sigma_{xx}\sigma_{yy}\left(2c^{\nu}_{1}-8g^{\nu}+2c^{\nu}_{2}\right)$$

and

$$\mathfrak{R}^{1\nu} = C^{\nu}_{ijkl}\sigma_{kl}\sigma_{ij},$$

it, according to above, is

$$\mathfrak{R}^{1\nu} = c^{\nu}_{1}\sigma^2_{xx}+4g^{\nu}\sigma^2_{xy}+c^{\nu}_{2}\sigma^2_{yy}+2c^{\nu}_{12}\sigma_{xx}\sigma_{yy}.$$

Thus

$$\mathfrak{S}^{4\nu} = 4\sigma^2_{xx}\left(c^{\nu}_{12}+2g^{\nu}-c^{\nu}_{1}\right)+8\sigma^2_{xy}\left(c^{\nu}_{2}-4g^{\nu}-2c^{\nu}_{12}+c^{\nu}_{1}\right)+4\sigma^2_{yy}\left(c^{\nu}_{12}+2g^{\nu}-c^{\nu}_{2}\right)+4\sigma_{xx}\sigma_{yy}\left(c^{\nu}_{2}-4g^{\nu}-2c^{\nu}_{12}+c^{\nu}_{1}\right).$$

Finally

$$\mathfrak{S}^{5\nu} = \mathfrak{R}^{2\nu},$$

and from above

$$\mathfrak{S}^{5\nu} = \sigma_{xx}\sigma_{xy}\left(8g^{\nu}+4c^{\nu}_{12}-4c^{\nu}_{1}\right)+\sigma_{yy}\sigma_{xy}\left(-8g^{\nu}-4c^{\nu}_{12}+4c^{\nu}_{2}\right).$$

Recapitulation of necessary condition

The necessary condition has form

$$\mathfrak{S}^{1\nu}\mathrm{tg}^4\alpha_\nu + \mathfrak{S}^{2\nu}\mathrm{tg}^3\alpha_\nu + \mathfrak{S}^{3\nu}\mathrm{tg}^2\alpha_\nu + \mathfrak{S}^{4\nu}\mathrm{tg}\,\alpha_\nu + \mathfrak{S}^{5\nu} = 0, \tag{16}$$

where

$$\mathfrak{S}^{1\nu} = 4\sigma_{xx}\sigma_{xy}\left(2g^\nu + c_{12}^\nu - c_2^\nu\right) + 4\sigma_{yy}\sigma_{xy}\left(-2g^\nu - c_{12}^\nu + c_1^\nu\right),$$

$$\mathfrak{S}_{2\nu} = \sigma_{xx}^2\left(4c_2^\nu - 4c_{12}^\nu - 8g^\nu\right) + \sigma_{xy}^2\left(32g^\nu - 8c_2^\nu + 16c_{12}^\nu - 8c_1^\nu\right) +$$

$$+\sigma_{yy}^2\left(4c_1^\nu - 4c_{12}^\nu - 8g^\nu\right) + \sigma_{xx}\sigma_{yy}\left(16g^\nu - 4c_2^\nu + 8c_{12}^\nu - 4c_1^\nu\right).$$

$$\mathfrak{S}^{3\nu} = 12\sigma_{xx}\sigma_{xy}\left(c_2^\nu - 2c_{12}^\nu - 4g^\nu + c_1^\nu\right) + 12\sigma_{yy}\sigma_{xy}\left(-c_2^\nu + 2c_{12}^\nu + 4g^\nu - c_1^\nu\right).$$

$$\mathfrak{S}^{4\nu} = 4\sigma_{xx}^2\left(c_{12}^\nu + 2g^\nu - c_1^\nu\right) + 8\sigma_{xy}^2\left(c_2^\nu - 4g^\nu - 2c_{12}^\nu + c_1^\nu\right) + 4\sigma_{yy}^2\left(c_{12}^\nu + 2g^\nu - c_2^\nu\right) + 4\sigma_{xx}\sigma_{yy}\left(c_2^\nu - 4g^\nu - 2c_{12}^\nu + c_1^\nu\right).$$

$$\mathfrak{S}^{5\nu} = \sigma_{xx}\sigma_{xy}\left(8g^\nu + 4c_{12}^\nu - 4c_1^\nu\right) + \sigma_{yy}\sigma_{xy}\left(-8g^\nu - 4c_{12}^\nu + 4c_2^\nu\right).$$

Solution of the problem

$$E_1^\nu = 181\,\mathrm{GPa}$$
$$E_2^\nu = 10{,}3\,\mathrm{GPa}$$
$$G_{12}^\nu = 7{,}17\,\mathrm{GPa}$$
$$\nu_{12}^\nu = 0{,}28$$

Table 1: Graphite-epoxy laminate ply material characteristics

For the cited material (Kevlar-Epoxy, vide Table 1) see following tables. There is dependance of searched winding angle with respect to given loading state σ_{xx}, σ_{yy} and σ_{xy}.

$\sigma_{yy} \lceil \sigma_{xy}$	0	10 MPa	20	30 MPa	40	50 MPa	60	70 MPa	80	90 MPa	100
0 MPa	0	-5.65	-10.90	-15.48	-19.33	-22.50	-25.01	-27.23	-29.00	-30.47	-31.72
10 MPa	0	-6.26	-11.98	-16.85	-20.82	-24.01	-26.57	-28.63	-30.32	-31.72	-32.89
20 MPa	0	-7.02	-13.28	-18.43	-22.50	-25.67	-28.15	-30.13	-31.72	-33.02	-34.10
30 MPa	0	-7.97	-14.87	-20.30	-24.41	-27.50	-29.87	-31.72	-33.19	-34.37	-35.35
40 MPa	0	-9.22	-16.85	-22.50	-26.57	-29.52	-31.72	-33.40	-34.72	-35.78	-36.65
50 MPa	0	-10.90	-19.33	-25.10	-29.00	-31.72	-33.70	-35.17	-36.32	-37.24	-37.98
60 MPa	0	-13.28	-22.50	-28.15	-31.72	-34.10	-35.78	-37.03	-37.98	-38.74	-39.35
70 MPa	0	-16.85	-26.57	-31.72	-34.72	-36.65	-37.98	-38.95	-39.69	-40.27	-40.73
80 MPa	0	-22.50	-31.72	-35.78	-37.98	-39.35	-40.27	-40.93	-41.44	-41.83	-42.14
90 MPa	0	-31.72	-37.98	-40.27	-41.44	-42.14	-42.62	-42.96	-43.21	-43.41	-43.57
100 MPa	-45	-45	-45	-45	-45	-45	-45	-45	-45	-45	-45

Table 2: Optimum winding angle α [°] versus loading stress $\sigma_{xx} = 100$ MPa and σ_{xy} [MPa], σ_{yy} [MPa] from table

$\sigma_{yy} \lceil \sigma_{xy}$	0	10 MPa	20 MPa	30 MPa	40 MPa	50 MPa	60 MPa	70 MPa	80 MPa	90 MPa	100
0	0	-10.90	-19.33	-25.10	-29.00	-31.72	-33.69	-35.17	-36.32	-37.24	-37.98
10 MPa	0	-13.28	-22.50	-28.15	-31.72	-34.10	-35.78	-37.03	-37.98	-38.74	-39.35
20 MPa	0	-16.85	-26.57	-31.72	-34.72	-36.65	-37.98	-38.95	-39.69	-40.27	-40.73
30 MPa	0	-22.50	-31.72	-35.78	-37.98	-39.35	-40.27	-40.93	-41.44	-41.83	-42.14
40 MPa	0	-31.72	-37.98	-40.27	-41.44	-42.14	-42.62	-42.96	-43.21	-43.41	-43.57
50 MPa	-45	-45	-45	-45	-45	-45	-45	-45	-45	-45	-45
60 MPa	-90	-58.28	-52.02	-49.73	-48.56	-47.86	-47.38	-47.04	-46.79	-46.59	-46.43
70 MPa	-90	-67.50	-58.28	-54.22	-52.02	-50.65	-49.73	-49.07	-48.56	-48.17	-47.855
80 MPa	-90	-73.15	-63.43	-58.28	-55.28	-53.35	-52.02	-51.05	-50.31	-49.73	-49.27
90 MPa	-90	-76.72	-67.50	-61.85	-58.28	-55.90	-54.22	-52.97	-52.02	-51.26	-50.65
100 MPa	-90	-79.10	-70.67	-64.90	-61.00	-58.28	-56.31	-54.83	-53.68	-52.76	-52.02

Table 3: Optimum winding angle α [°] versus loading stress $\sigma_{xx} = 50$ MPa and σ_{xy} [MPa], σ_{yy} [MPa] from table

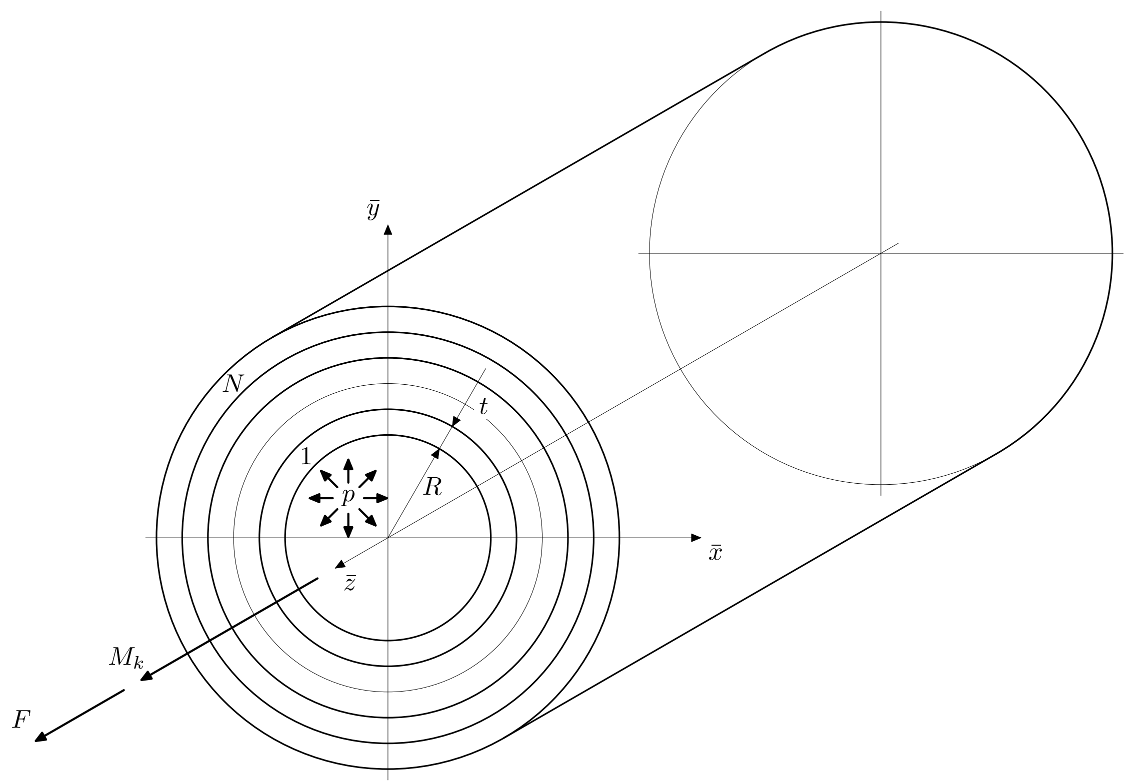


Fig. 12: Multilayer laminate tube under tension, torsion and interior pressure (it is $t \ll R$)

Multilayer laminate tube under tension, torsion and interior pressure

With the above asumptions[7] we have again the problem

$$\{\hat{\boldsymbol{C}}, \hat{\boldsymbol{\sigma}}\} = \arg\min_{\boldsymbol{C}\in\mathbb{C}} \min_{\boldsymbol{\sigma}\in\mathbb{S}^N} \int_\Omega C_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}\Omega,$$

[7]See section on page 39.

where $\mathbb{C}$ is set of all admissible compliance tensors and

$$\mathbb{S}^N = \left\{ \sigma^\nu_{xx}, \sigma^\nu_{yy}, \sigma^\nu_{xy} \ (\nu = 1, 2, \ldots, N) \mid F = \sum_{\nu=1}^{N} A^\nu \sigma^\nu_{xx}, \ \frac{pR}{t} = \sum_{\nu=1}^{N} \sigma^\nu_{yy}, \ M_k = \sum_{\nu=1}^{N} S^\nu \sigma^\nu_{xy} \right\}$$

where N is number of plies, t is ply thickness (the tube thickness is tN), R is inner radius, A^ν is cross-section of νth ply:

$$A^\nu = \pi \left((R + \nu t)^2 - (R + (\nu - 1)t)^2 \right) \tag{17}$$

and S^ν is static moment of the same cross-section:

$$S^\nu = \frac{2\pi}{3} \left((R + \nu t)^3 - (R + (\nu - 1)t)^3 \right).$$

Objective functional

$$\mathfrak{c} = \int_\Omega C_{ijkl} \sigma_{ij} \sigma_{kl} \, \mathrm{d}\Omega$$

is arranged as follows

$$\mathfrak{c} = l \int_R^{R+tN} C_{ijkl} \sigma_{ij} \sigma_{kl} 2\pi \varrho \, \mathrm{d}\varrho = 2\pi l \sum_{\nu=1}^{N} \int_{R+(\nu-1)t}^{R+\nu t} C_{ijkl} \sigma_{ij} \sigma_{kl} \varrho \, \mathrm{d}\varrho.$$

From section *Single Ply Laminate Tube under Tension, Torsion and Interior Pressure* we have for $\varrho \in \langle R + (\nu - 1)t, R + \nu t \rangle$ on p. 39 relation (12)

$$C_{ijkl} \sigma_{ij} \sigma_{kl} = \left(\mathcal{R}^{1\nu}_{ijkl} c^4_\nu + \mathcal{R}^{2\nu}_{ijkl} c^3_\nu s_\nu + \mathcal{R}^{3\nu}_{ijkl} c^2_\nu s^2_\nu + \mathcal{R}^{4\nu}_{ijkl} c_\nu s^3_\nu + \mathcal{R}^{5\nu}_{ijkl} s^4_\nu \right) \sigma^\nu_{ij} \sigma^\nu_{kl}$$

and thus

$$\mathfrak{c} = l \sum_{\nu=1}^{N} A^\nu \left(\mathcal{R}^{1\nu}_{ijkl} c^4_\nu + \mathcal{R}^{2\nu}_{ijkl} c^3_\nu s_\nu + \mathcal{R}^{3\nu}_{ijkl} c^2_\nu s^2_\nu + \mathcal{R}^{4\nu}_{ijkl} c_\nu s^3_\nu + \mathcal{R}^{5\nu}_{ijkl} s^4_\nu \right) \sigma^\nu_{ij} \sigma^\nu_{kl},$$

where[8]

$$A^\nu = \pi \left((R + \nu t)^2 - (R + (\nu - 1)t)^2 \right)$$

and constants $\mathcal{R}^{\varrho\nu}_{ijkl}$ are above.

It is convenient with regards to searching the stacionary point to take the fact $\sigma^\nu_{xy} = \sigma^\nu_{yx}$ and write the objective function as

$$\mathfrak{c} = \frac{1}{2} \sum_{\nu=1}^{N} A^\nu \boldsymbol{\sigma}^{\nu T} \boldsymbol{P}^\nu \boldsymbol{\sigma}^\nu,$$

where

$$\boldsymbol{\sigma}^\nu = \begin{pmatrix} \sigma^\nu_{xx} \\ \sigma^\nu_{yy} \\ \sigma^\nu_{xy} \end{pmatrix}$$

and

$$\boldsymbol{P}^\nu = \boldsymbol{R}^{1\nu} c^4_\nu + \boldsymbol{R}^{2\nu} c^3_\nu s_\nu + \boldsymbol{R}^{3\nu} c^2_\nu s^2_\nu + \boldsymbol{R}^{4\nu} c_\nu s^3_\nu + \boldsymbol{R}^{5\nu} s^4_\nu, \tag{18}$$

$$\boldsymbol{R}^{1\nu} = \begin{pmatrix} c^\nu_1 & c^\nu_{12} & 0 \\ c^\nu_{12} & c^\nu_2 & 0 \\ 0 & 0 & 4g^\nu \end{pmatrix},$$

$$\boldsymbol{R}^{2\nu} = \begin{pmatrix} 0 & 0 & -2c^\nu_1 + 2c^\nu_{12} + 4g^\nu \\ 0 & 0 & 2c^\nu_2 - 2c^\nu_{12} - 4g^\nu \\ -2c^\nu_1 + 2c^\nu_{12} + 4g^\nu & 2c^\nu_2 - 2c^\nu_{12} - 4g^\nu & 0 \end{pmatrix},$$

[8] See (17) on p. 45.

$$\boldsymbol{R}^{3\nu} = \begin{pmatrix} 2c_{12}^\nu + 4g^\nu & c_1^\nu + c_2^\nu - 4g^\nu & 0 \\ c_1^\nu + c_2^\nu - 4g^\nu & 2c_{12}^\nu + 4g^\nu & 0 \\ 0 & 0 & 4c_1^\nu + 4c_2^\nu - 8c_{12}^\nu - 8g^\nu \end{pmatrix},$$

$$\boldsymbol{R}^{4\nu} = \begin{pmatrix} 0 & 0 & 2c_2^\nu - 2c_{12}^\nu - 4g^\nu \\ 0 & 0 & -2c_1^\nu + 2c_{12}^\nu + 4g^\nu \\ 2c_2^\nu - 2c_{12}^\nu - 4g^\nu & -2c_1^\nu + 2c_{12}^\nu + 4g^\nu & 0 \end{pmatrix},$$

$$\boldsymbol{R}^{5\nu} = \begin{pmatrix} c_2^\nu & c_{12}^\nu & 0 \\ c_{12}^\nu & c_1^\nu & 0 \\ 0 & 0 & 4g^\nu \end{pmatrix}.$$

Let us solve the attained form of our problem

$$\{\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\sigma}}\} = \arg \min_{\boldsymbol{\alpha}} \min_{\boldsymbol{\sigma} \in \mathbb{S}^N} \mathfrak{c},$$

$$\mathbb{S}^N = \left\{ \boldsymbol{\sigma} \mid \sum_{\nu=1}^{N} \boldsymbol{S}^\nu \boldsymbol{\sigma}^\nu = \boldsymbol{F} \right\}$$

with

$$\boldsymbol{S}^\nu = \begin{pmatrix} A^\nu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S^\nu \end{pmatrix} \quad \boldsymbol{F} = \begin{pmatrix} F \\ \frac{pR}{t} \\ M_k \end{pmatrix}$$

by Lagrange theorem. The items of this theorem at discused problem are

a)

$$\mathfrak{L}_N = \lambda_0 \frac{1}{2} \sum_{\nu=1}^{N} A^\nu \boldsymbol{\sigma}^{\nu T} \boldsymbol{R}^\nu \boldsymbol{\sigma}^\nu + \boldsymbol{\lambda}^T \left(\sum_{\nu=1}^{N} \boldsymbol{S}^\nu \boldsymbol{\sigma}^\nu - \boldsymbol{F} \right),$$

where

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

and

$$\frac{\partial \mathfrak{L}_N}{\partial \boldsymbol{\sigma}^\mu} = \lambda_0 A^\mu \boldsymbol{P}^\mu \boldsymbol{\sigma}^\mu + \boldsymbol{S}^\mu \boldsymbol{\lambda} = \boldsymbol{0} \quad (\mu = 1, 2, \ldots, N), \tag{19}$$

$$\frac{\partial \mathfrak{L}_N}{\partial \alpha_\mu} = \frac{\lambda_0}{2} A^\mu \boldsymbol{\sigma}^{\mu T} \frac{\partial \boldsymbol{P}^\mu}{\partial \alpha_\mu} \boldsymbol{\sigma}^\mu = 0 \quad (\mu = 1, 2, \ldots, N), \tag{20}$$

where

$$\frac{\partial \boldsymbol{P}^\mu}{\partial \alpha^\mu} = -4\boldsymbol{R}^{1\mu} c_\mu^3 s_\mu + \boldsymbol{R}^{2\mu} \left(-3c_\mu^2 s_\mu^2 + c_\mu^4 \right) + \boldsymbol{R}^{3\mu} \left(-2c_\mu s_\mu^3 + 2c_\mu^3 s_\mu \right) + \boldsymbol{R}^{4\mu} \left(-s_\mu^4 + 3c_\mu^2 s_\mu^2 \right) + 4\boldsymbol{R}^{5\mu} s_\mu^3 c_\mu.$$

b) $\lambda_0 \geq 0$. It is clear from (20) that $\lambda_0 \neq 0$, and thus let us say $\lambda_0 = 1$.

c) Complementary condition is fulfil by keeping constrains

$$\sum_{\nu=1}^{N} \boldsymbol{S}^\nu \boldsymbol{\sigma}^\nu = \boldsymbol{F}. \tag{21}$$

For the solution of system of neccesary conditions (19), (20), (21) we use the method of alternating fulfilment of necessary conditions:[9]

[9] Srvn. [Allaire, 2002].

1. We choose α_μ ($\mu = 1, 2, \ldots, N$), e.g. $\alpha_\mu = 0\ \forall\mu$.

2. For given set of winding angles α_μ ($\mu = 1, 2, \ldots, N$) we solve elasticity problem that is formulated by the system of equations (19) and (21) that are linear with respect to σ^μ_{xx}, σ^μ_{yy}, σ^μ_{xy} ($\mu = 1, 2, \ldots, N$), $\boldsymbol{\lambda}$.

3. For this stresses $\boldsymbol{\sigma}^\mu$ ($\mu = 1, 2, \ldots, N$) let us solve equations (20) at α_μ ($\mu = 1, 2, \ldots, N$). This solution we again use at item 2. Etc. to convergence.[10]

Solutiom of problem from item 2

We solve the system of equations (19), (21), where $\lambda_0 = 1$ and α_μ ($\mu = 1, 2, \ldots, N$) are given. We have

$$\boldsymbol{\sigma}^\mu = -\left(A^\mu \boldsymbol{P}^\mu\right)^{-1} \boldsymbol{S}^\mu \boldsymbol{\lambda} \quad (\mu = 1, 2, \ldots, N),$$

where along (21)

$$\sum_{\nu=1}^{N} \boldsymbol{S}^\nu \left(A^\nu \boldsymbol{P}^\nu\right)^{-1} \boldsymbol{S}^\nu \boldsymbol{\lambda} = -\boldsymbol{F},$$

thus

$$\boldsymbol{\lambda} = -\left(\sum_{\nu=1}^{N} \boldsymbol{S}^\nu \left(A^\nu \boldsymbol{P}^\nu\right)^{-1} \boldsymbol{S}^\nu\right)^{-1} \boldsymbol{F}$$

and finally

$$\boldsymbol{\sigma}^\mu = \left(A^\mu \boldsymbol{P}^\mu\right)^{-1} \boldsymbol{S}^\mu \left(\sum_{\nu=1}^{N} \boldsymbol{S}^\nu \left(A^\nu \boldsymbol{P}^\nu\right)^{-1} \boldsymbol{S}^\nu\right)^{-1} \quad (\mu = 1, 2, \ldots, N).$$

Solution of problem from item 3

Let us solve the system of independent equations

$$H^{1\nu}\mathrm{tg}^4\alpha + H^{2\nu}\mathrm{tg}^3\alpha + H^{3\nu}\mathrm{tg}^2\alpha + H^{4\nu}\mathrm{tg}\,\alpha + H^{5\nu} = 0 \quad (\nu = 1, 2, \ldots, N), \tag{22}$$

where

$$\begin{aligned} H^{1\nu} &= \boldsymbol{\sigma}^{\nu T}\left(-\boldsymbol{R}^{4\nu}\right)\boldsymbol{\sigma}^\nu, \\ H^{2\nu} &= \boldsymbol{\sigma}^{\nu T}\left(4\boldsymbol{R}^{5\nu} - 2\boldsymbol{R}^{3\nu}\right)\boldsymbol{\sigma}^\nu, \\ H^{3\nu} &= \boldsymbol{\sigma}^{\nu T}\left(3\boldsymbol{R}^{4\nu} - 3\boldsymbol{R}^{2\nu}\right)\boldsymbol{\sigma}^\nu, \\ H^{4\nu} &= \boldsymbol{\sigma}^{\nu T}\left(2\boldsymbol{R}^{3\nu} - 4\boldsymbol{R}^{1\nu}\right)\boldsymbol{\sigma}^\nu, \\ H^{5\nu} &= \boldsymbol{\sigma}^{\nu T}\boldsymbol{R}^{2\nu}\boldsymbol{\sigma}^\nu. \end{aligned}$$

It is clear that this problem have more that one solution. From all real solutions we choose that which minimize the objective functinon

$$\mathfrak{c} = \frac{1}{2}\sum_{\nu=1}^{N} A^\nu \boldsymbol{\sigma}^{\nu T}\boldsymbol{P}^\nu\boldsymbol{\sigma}^\nu \tag{23}$$

Such piont is global minimum point.

Examples

For material from tab. 1 on p. 43 and given loading we have obtained by the above described method the following optimum winding angles where $R = 10\,\mathrm{mm}$ and $t = 1\,\mathrm{mm}$.

[10] [Allaire, 2002] stated: this method converge.

F [N]	p [MPa]	M_k [Nmm]	α [°]	σ_{xx} [MPa]	σ_{yy} [MPa]	σ_{xy} [MPa]	cpu [s]
628	0	0	0	9,52	0	0	0,07
0	1	0	$\pm 90°$	0	10	0	0,09
0	0	5 500	$\pm 45°$	0	0	7,93	0,1
628	1	0	$\pm 90°$	9,52	10	0	0,08
800	1	0	$0°$	12,13	10	0	0,12
628	0	5 500	$-29,52°$	9,52	0	7,93	0,42
0	1	5 500	$-61,11°$	0	10	7,93	0,08
628	1	5 500	$-45,87°$	9,52	10	7,93	0,1
628	1	-5 500	$45,87°$	9,52	10	-7,93	0,1

Table 4: Single Ply Laminate Tube under Tension, Torsion and Interior Pressure ($N = 1$)

F [N]	p [MPa]	M_k [Nmm]	α_1 [°]	α_2 [°]	σ^1_{xx} [MPa]	σ^1_{yy}	σ^1_{xy}	σ^2_{xx}	σ^2_{yy}	σ^2_{xy}	cpu [s]
628	0	0	0	0	4,54	0	0	4,54	0	0	0,12
0	1	0	$\pm 90°$	$\pm 90°$	0	5,23	0	0	4,77	0	0,12
0	0	5 500	$-44,4°$	$45,6°$	3,42	3,27	3,63	-3,13	-3,27	3,59	2,88
0	0	-5 500	$-44,4°$	$45,6°$	-3,42	-3,27	-3,63	3,13	3,27	-3,59	1,81
628	1	0	$\pm 90°$	0	0,78	9,21	0	7,98	0,79	0	0,13
628	0	5 500	$61,93°$	$-29,34°$	-0,28	-2,80	1,87	8,95	2,80	5,05	0,99
0	1	5 500	$24,90°$	$-62,65°$	-2,90	0,033	1,74	2,65	9,97	5,17	1,02
628	1	5 500	$-59,07°$	$-34,57°$	2,92	6,75	3,57	6,02	3,25	3,63	4,27
628	1	-5 500	$59,07°$	$34,57°$	2,92	6,75	-3,57	6,02	3,25	-3,63	3,88

Table 5: Two Ply Laminate Tube under Tension, Torsion and Interior Pressure ($N = 2$)

F [N]	p [MPa]	M_k [Nmm]	α_1 [°]	α_2 [°]	α_3 [°]	cpu [s]
628	0	0	0	0	0	0,28
0	1	0	$\pm 90°$	$\pm 90°$	$\pm 90°$	0,36
0	0	5 500	$-43{,}79°$	$-44{,}15°$	$45{,}54°$	1,72
0	0	-5 500	$-43{,}79°$	$-44{,}15°$	$45{,}54°$	1,70
628	1	0	$\pm 90°$	$\pm 90°$	0	0,29
628	0	5 500	$63{,}2°$	$-28{,}38°$	$-29{,}69°$	4,23
628	0	-5 500	$-63{,}2°$	$28{,}38°$	$29{,}69°$	4,28
0	1	5 500	$-65{,}87°$	$26{,}37°$	$-61{,}69°$	2,55
628	1	5 500	$-66{,}81°$	$-48{,}43°$	$-28{,}11°$	24,48
628	1	-5 500	$66{,}81°$	$48{,}43°$	$28{,}11°$	24,37
-628	1	-5 500	$74{,}13°$	$72{,}48°$	$-19{,}05°$	0,65

Table 6: Three Ply Laminate Tube under Tension, Torsion and Interior Pressure ($N = 3$)

Design of laminate tube under uncertain loading

Common rules

Let us consider a problem to search laminate tube winding angle that maximize the stiffness at the most danger loading from the given set of possible loading.

Generally we have the problem to search a compliance tensor $\hat{\boldsymbol{C}} \in \mathbb{C}$ that minimize compliance measure $\mathfrak{l} = \mathfrak{l}(\hat{\boldsymbol{\sigma}})$ at the case when $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\hat{\boldsymbol{t}})$ is the stress state developed by the worst configuration of loading $\hat{\boldsymbol{t}} \in \mathbb{T}$ from a set of possible loading $\mathbb{T}$:

$$\hat{\boldsymbol{C}} = \arg\min_{\boldsymbol{C}\in\mathbb{C}} \max_{\boldsymbol{t}\in\mathbb{T}} \mathfrak{l}(\boldsymbol{u}(\hat{\boldsymbol{\sigma}}(\boldsymbol{t}))), \tag{24}$$

i.e.,

$$\{\hat{\boldsymbol{C}}, \hat{\boldsymbol{t}}, \hat{\boldsymbol{\sigma}}\} = \arg\min_{\boldsymbol{C}\in\mathbb{C}} \max_{\boldsymbol{t}\in\mathbb{T}} \min_{\boldsymbol{\sigma}\in\mathbb{S}} \int_\Omega C_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}\Omega,$$

where $\mathbb{C}$ is the set of acceptable compliance tensors, $\mathbb{T}$ is the set of possible load states, and $\mathbb{S}$ is the set of balanced stress states

$$\mathbb{S} = \{\sigma_{ij} \mid \sigma_{ij,i} + p_j = 0 \text{ in } \Omega \;\wedge\; \sigma_{ij}\ell_j = t_i \text{ on } \partial_t\Omega\}\,.$$

Formalizing of the problem

At the considered case of one-ply tube under tension, torsion and interior pressure the set $\mathbb{S}$ has one point:

$$\mathbb{S} = \left\{\sigma_{xx} = \frac{N}{2\pi Rt},\ \sigma_{yy} = \frac{pR}{t},\ \sigma_{xy} = \frac{M_k}{2\pi R^2 t}\right\}.$$

Thus the inner minimization is dissolved. Let us choose the set $\mathbb{T}$ such that

$$\mathbb{T} = \left\{N, M_k, p \mid \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{xy}^2 = S\right\},$$

where the S is given constant. From that, with the above manipulation, we have problem

$$\left\{\hat{\alpha}, \hat{\boldsymbol{\sigma}}^t\right\} = \arg\min_{\alpha_\nu} \max_{\sigma_{ij}\in\mathbb{T}^\sigma} \mathfrak{c}, \tag{25}$$

$$\mathfrak{c} = \frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{P}^\nu \boldsymbol{\sigma},$$

where

$$\mathbb{T}^\sigma = \left\{\boldsymbol{\sigma} \mid \boldsymbol{\sigma}^T\boldsymbol{\sigma} = S\right\}$$

and $\boldsymbol{P}^\nu$ is given by (18) at p. 45. Stress $\boldsymbol{\sigma}$ is the same as stress $\boldsymbol{\sigma}^\nu$ ($\nu = 1$) from above.

The problem (25) is solved by consecutive resolution of inner and outer problems:

1. The inner maximize problem
$$\hat{\boldsymbol{\sigma}}(\alpha_\nu) = \arg\max_{\boldsymbol{\sigma}\in\mathbb{T}^\sigma} \mathfrak{c}(\alpha_\nu, \boldsymbol{\sigma}). \tag{26}$$

2. The outer problem, where at item 1 searched stress depending on winding angle $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\alpha_\nu)$ is substituted into (25), have form
$$\hat{\alpha} = \arg\min_{\alpha_\nu} \mathfrak{c}(\alpha_\nu, \boldsymbol{\sigma}(\alpha_\nu)).$$

The last two problems we solve by using (above cited) Lagrange theorem.

Resolution of the problem from item 1 (stress)

We solve the problem

$$\hat{\boldsymbol{\sigma}} = \arg\max_{\boldsymbol{\sigma}\in\mathbb{T}^\sigma} \mathfrak{c}(\alpha_\nu, \boldsymbol{\sigma}).$$

According to Lagrange Theorem we have:

a) Stacionarity condition of Lagrange function

$$\mathfrak{L}_c = \frac{\lambda_0}{2}\boldsymbol{\sigma}^T \boldsymbol{P}^\nu \boldsymbol{\sigma} + \lambda(\boldsymbol{\sigma}^T\boldsymbol{\sigma} - S)$$

have form

$$\frac{\partial \mathfrak{L}_c}{\partial \boldsymbol{\sigma}} = \mathbf{0},$$

i.e.,

$$\lambda_0 \boldsymbol{P}^\nu \boldsymbol{\sigma} + 2\lambda\boldsymbol{\sigma} = \mathbf{0}, \tag{27}$$

where the point $\boldsymbol{\sigma}$ must meet the condition

$$\boldsymbol{\sigma}^T\boldsymbol{\sigma} = S. \tag{28}$$

b) Condition of sign inverse correspondence is reduced to

$$\lambda_0 \le 0,$$

as follows from interchange of maximizing the function $\mathfrak{c}$ and minimizing the function $-\mathfrak{c}$.

c) Complementarity condition is fulfilled *á priori*.

If $\lambda_0 \neq 0$, then it is possible to choose it arbitralily (but whit respect to item b).[11] Let us consider $\lambda_0 \neq 0$ and choose $\lambda_0 = -1$. Thus the coefficient λ is determined by constraint (28).

Stacionarity condition (27) takes the form

$$\boldsymbol{P}^\nu \boldsymbol{\sigma} - \ell\boldsymbol{\sigma} = \mathbf{0},$$

where we write $\ell = 2\lambda$.

This is system of linear equations at variable σ_{ij}

$$\boldsymbol{V}\boldsymbol{\sigma} = \mathbf{0}, \tag{29}$$

where

$$\boldsymbol{V} = \boldsymbol{P}^\nu - \ell\boldsymbol{I}, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This problem has notrivial solution only if ℓ is eigenvalue of matrix $\boldsymbol{P}$. We first search eigenvalues ℓ^i a eigenvectors $\boldsymbol{s}^i$ $(i = 1, 2, 3)$ of matrix $\boldsymbol{R}$. Only real solution is interesting for us. Then we search the solution of neccesary condition at the form

$$\boldsymbol{\sigma} = t^i \boldsymbol{s}^i,$$

where $t^i \in \mathbb{R}$ we determine from condition

$$\sigma_{xx}^2 + \sigma_{xy}^2 + \sigma_{yy}^2 = S.$$

From that

$$t^i = \sqrt{\frac{S}{s_1^2 + s_2^2 + s_3^2}}.$$

The value of the objective function (25) determines the true solution.

For a specified value of winding angle the solution searching of this problem is very easy. On the other hand common solution and analytical representation of dependance $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\alpha_\nu)$ is extremely difficult to find. This is the case when it is appropriate to use the *response surface method*:[12]

[11] See [LAVRENŤJEV and LUSTERNIK, 1952] and [ALEXEJEV *et al.*, 1991].
[12] See e.g. [VENTER *et al.*, 1996] and [VENTER and HAFTKA, 1997].

i) For values $\alpha^k = -\frac{\pi}{2} + \frac{k\pi}{N} \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$ $(k = 0, 1, 2, \ldots, N)$ we search relevant value of dangerous stress $\boldsymbol{\sigma}$ from above relations.

ii) By a least square method we then search coefficients of elected expression that express this stress. This expression must meet two conditions: Firstly, to allow satisfactory accuracy, and secondly, to be advisable and applicable for solution of the problem from item b.

Let us observe accuacy of following model (applicability for solution have led to his selection)

$$\boldsymbol{\sigma} = \boldsymbol{a}_p \cos^p \alpha^k + \boldsymbol{a}_{p-1} \cos^{p-1} \alpha^k \sin \alpha^k + \ldots + \boldsymbol{a}_1 \cos \alpha^k \sin^{p-1} \alpha^k + \boldsymbol{a}_0 \sin^p \alpha^k \quad (\boldsymbol{a}_p \in \mathbb{R}^3, p \in \mathbb{N}).$$

Formula for searching of coefficients $\boldsymbol{a}_i$ – least square method

For given p our problem contains $3 \times p$ unknowns $\boldsymbol{a}_p \in \mathbb{R}^3$ $(i = 1, 2, \ldots, p)$. For one solution of numerical experiment (for elected winding angle α_k) we have system of equations

$$\boldsymbol{\sigma}^k = \boldsymbol{A}^k \boldsymbol{a} = \begin{pmatrix} \boldsymbol{I}_{3\times3} \cos^p \alpha^k, & \boldsymbol{I}_{3\times3} \cos^{p-1} \alpha^k \sin \alpha^k, & \ldots, & \boldsymbol{I}_{3\times3} \cos \alpha^k \sin^{p-1} \alpha^k, & \boldsymbol{I}_{3\times3} \sin^p \alpha^k \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_p \\ \boldsymbol{a}_{p-1} \\ \vdots \\ \boldsymbol{a}_0 \end{pmatrix}.$$

For N solutions of numerical experiment it then is

$$\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{a},$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma}^0 \\ \boldsymbol{\sigma}^1 \\ \vdots \\ \boldsymbol{\sigma}^N \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}^0 \\ \boldsymbol{A}^1 \\ \vdots \\ \boldsymbol{A}^N \end{pmatrix}.$$

At the least square method sence it is

$$\boldsymbol{a} = \left(\boldsymbol{A}^T \boldsymbol{A}\right)^{-1} \boldsymbol{A}^T \boldsymbol{\Sigma}.$$

Neccesary condition from second item (winding angle)

Neccesary condition for problem from second item

$$\frac{\partial \mathfrak{c}(\alpha, \boldsymbol{\sigma}(\alpha))}{\partial \alpha} = 0$$

is determined by objective function expression

$$\mathfrak{c}(\alpha, \boldsymbol{\sigma}(\alpha)) = \sum_{i=0}^{p} \boldsymbol{a}_i^T \cos^i \alpha \sin^{p-i} \alpha \left(\boldsymbol{R}^{1\nu} \cos^4 \alpha + \boldsymbol{R}^{2\nu} \cos^3 \alpha \sin \alpha + \boldsymbol{R}^{3\nu} \cos^2 \alpha \sin^2 \alpha + \right.$$

$$\left. + \boldsymbol{R}^{4\nu} \cos \alpha \sin^3 \alpha + \boldsymbol{R}^{5\nu} \sin^4 \alpha \right) \sum_{j=0}^{p} \boldsymbol{a}_j^T \cos^j \alpha \sin^{p-j} \alpha,$$

or better

$$\mathfrak{c} = \sum_{i,j=0}^{p} \sum_{\varrho=1}^{5} \boldsymbol{a}_i^T \boldsymbol{R}^{\varrho\nu} \boldsymbol{a}_j \cos^{i+5-\varrho+j} \alpha \sin^{2p-i-j+\varrho-1} \alpha.$$

Thus the neccesary condition has the form

$$\frac{\partial \mathfrak{c}}{\partial \alpha} = \sum_{i,j=0}^{p} \sum_{\varrho=1}^{5} \boldsymbol{a}^T \boldsymbol{R}^{\varrho\nu} \boldsymbol{a}_j \left(-(i + 5 - \varrho + j) \cos^{i+4-\varrho+j} \alpha \sin^{2p-i-j+\varrho} \alpha + \right.$$

$$\left. + (2p - i - j + \varrho - 1) \cos^{i+6-\varrho+j} \alpha \sin^{2p-i-j+\varrho-2} \alpha \right) = 0.$$

Example

For material characteristics from tab. 1 on p. 43 and $S = 10000$ MPa^2 we have the following:

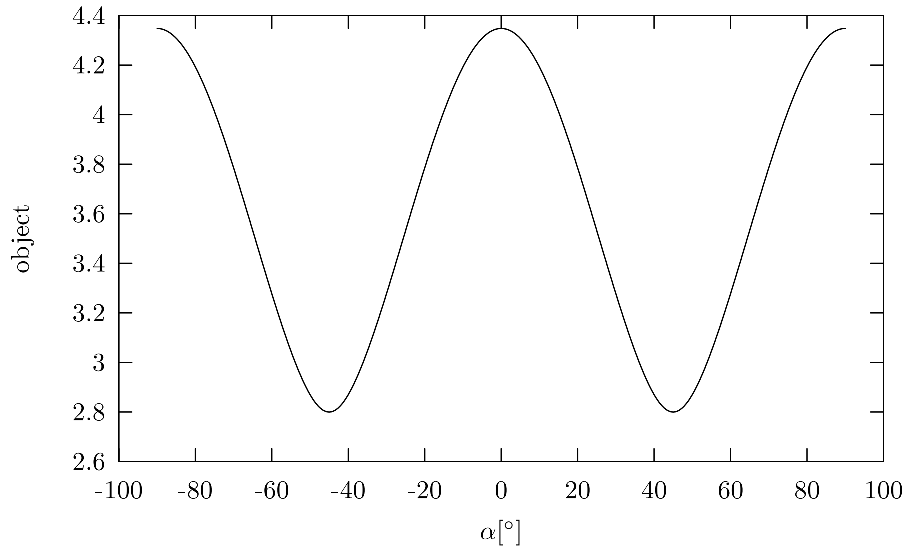


Fig. 13: Objectiv function for dangerous stress vs. winding angle α

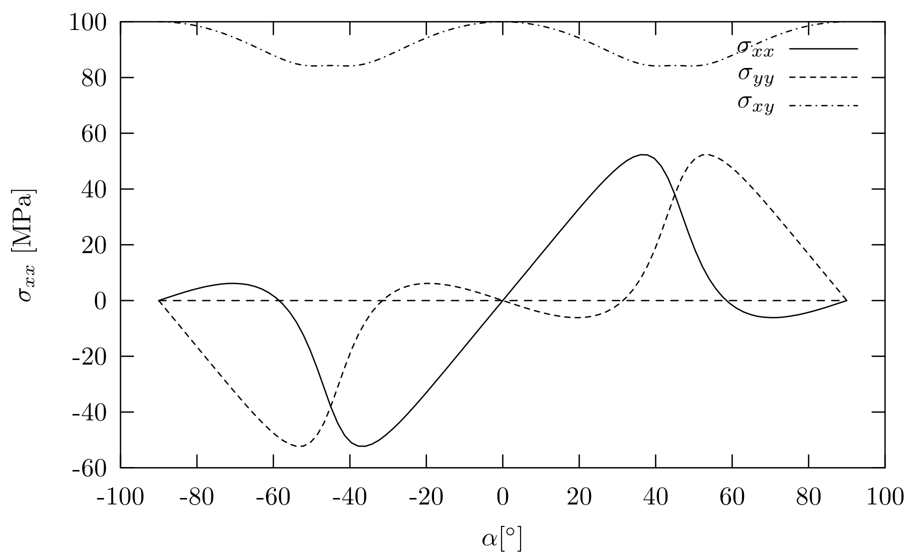
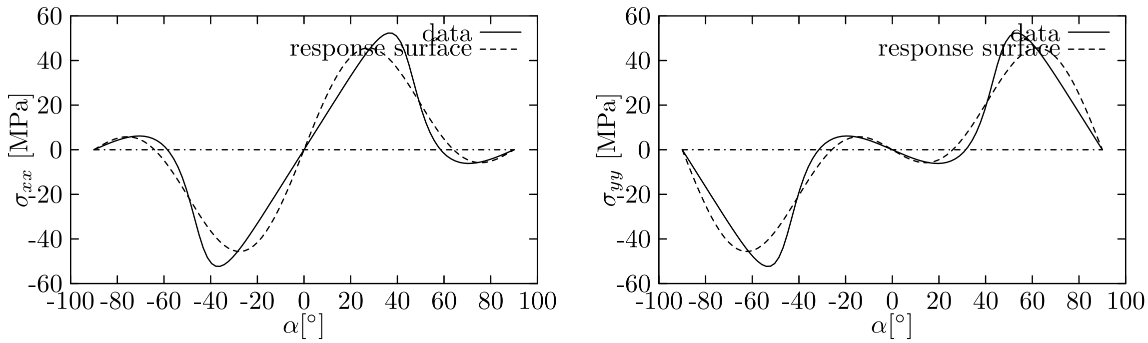


Fig. 14: Dangerous stress $\boldsymbol{\sigma}$ vs. winding angle α

Comparison of response surface with input data

Let us compare the response surface with input data (for $p = 4$) by grafic wiew.



(a) Dangerous stress σ_{xx} (b) Dangerous stress σ_{yy}

Fig. 15: Dangerous stress σ_{xx} and σ_{yy} vs. winding angle α

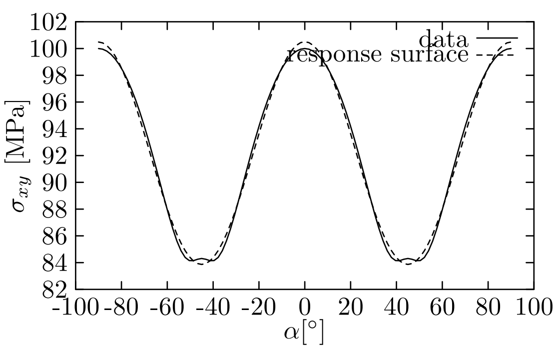


Fig. 16: Dangerous stress σ_{xy} vs. winding angle α

Completion of problem solution – outer optimization

From above it is clear that the minimum value of ojective function for the most dangerous loading is achieved at $\pm 45°$ of the winding angle.

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