# Dynamical Analysis of Elastic Rod: Example of Using Free Software

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***Abstract*—We present an original method for semi-analytical solution of elastic body dynamics. Towards this end, we use the principle of least action, tensor calculus, curvilinear elasticity, and Fourier series expansion. Our method is applicable not only for the straight rods but also to curved rods on curved surfaces, and, utilizing the Lagrange's multiplier method, also for elastic multibody systems.**

***Index Terms*—curvilinear elasticity, dynamics of elastic bodies, tensor calculus, priciple of least action, free software (GPL)**

## I. Introduction

There are several usually used approaches for the solution of the problem of deformable body dynamics, such as incremental finite element method, finite segment method, large rotation vector, linear theory of elastodynamics, and so on; see, e.g., [2],[7],[3],[1]. For detailed discussion on the survey of these traditional methods, see [8].

Nevertheless, our approach is based on even more traditional concepts and tools, such as the principle of least action [6], tensor calculus [5], curvilinear elasticity [4], and Fourier series expansion. In first sections we describe the methodology of our approach. Later, we specify our approach analyzing the single elastic rod dynamics problem and, in relevant figures at the end, we present the results of the simulations implemented using GNU[1] and other free software packages distributed under GPL (General Public License).

## II. The illustrating problem

We analyze a straight rod of the length $l$ and rectangular cross-section with constant thickness, $w$, and constant height, $2h$, according to the Figure 1. The rod is connected to the frame in the point $O$ via rotational joint. We introduce the following coordinate systems: a) the global Cartesian coordinate system, $b$, with axes $b^a$ $(a = 1, 2, 3)$ fixed with the frame, b) the space Cartesian coordinate system, $x$, with axes $x^a$ that rotates with respect to the system $b$ about the axis $b^3$, and c) the material curvilinear coordinate system, $\xi$, with axes $\xi^a$. The coordinate system $\xi$ before the deformation coincides with the system $x$. The system $x$ is introduced so that $x^3 \equiv b^3 \equiv \xi^3$.

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[1] `http://www.gnu.org/`

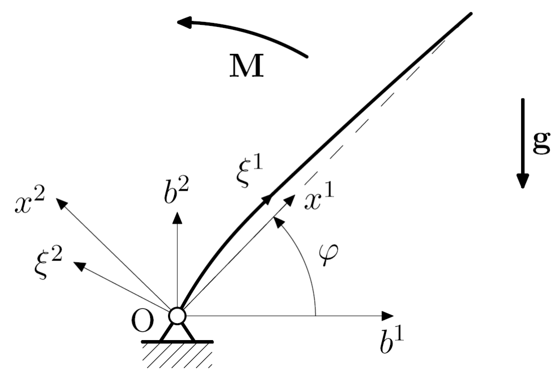


Fig. 1. The elastic rod.

As the dynamical analysis is based on the principle of lest action we are going to express the kinetic and elastic energy of the element of the rod. The integration with respect to the volume of the rod shall be carried out in the system $\xi$.

We consider the external moment acting on the rod applied at the point $O$ and having the direction of the axis $b^3$. We also concider the gravitational field having the direction of $-b^2$.

## III. Principle of least action

According to the principle of least action the minimizer of the action

$$S = \int_{t_0}^{t} L\mathrm{d}t = \int_{t_0}^{t} (T - \Pi)\mathrm{d}t, \tag{1}$$

where $T$ is the kinetic energy of the rod, $\Pi$ is the total potential energy of the rod that is given as the sum of elastic energy, $U$, potential energy of gravitational force field, $Z$, and potential energy of external forces, $-W$, solves the problem of dynamical analysis.

The dependant variable of this variational problem is the spatial configuration (i.e. the position and the shape) of the deformed rod in the space $b$. The position is given by the set of the following coordinates: $\varphi$ - rotation of the system $x$ with respect to the system $b$, $v$ - the displacement of points of the rod's longitudinal axis (centreline) of the deformed rod expressed in the system $x$. These dependant variables are functions of independent variables $t$ (time), $\xi^a$ (material coordinates).

In accordance with the defined problem we consider the deformation only in the $(x^1,x^2)$ plane.

The displacement $v$ is approximated via Fourier series expansion with the base functions given by the solution of

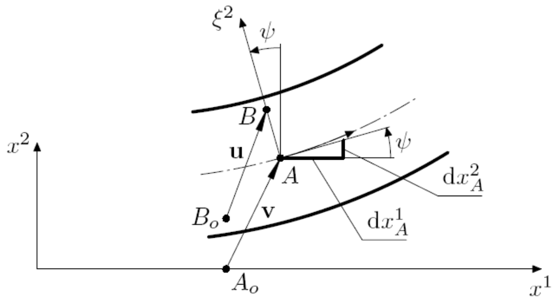


Fig. 2. The point $A$ of the centreline and a general point $B$ of the section before (index 0) and after the deformation.

the eigenproblem of both the longitudinal and the transversal vibration of the rod satisfying the boundary conditions of a rod simply supported at the both ends

$$K_k^1 \equiv \sin(\frac{\pi\xi^1}{2l}(2k-1)) \qquad (k = 1, 2, \ldots), \tag{2}$$

$$K_k^2 \equiv \sin(\frac{\pi k\xi^1}{l}) \qquad (k = 1, 2, \ldots). \tag{3}$$

## IV. Elastic energy

For the elastic energy we have

$$U = \frac{1}{2}\int\limits_{(V)} \bar{E}^{klmn}\bar{\gamma}_{kl}\bar{\gamma}_{mn}\,\mathrm{d}V, \tag{4}$$

where

$$2\bar{\gamma}_{ab} = \breve{g}_{ab} - \bar{g}_{ab} \tag{5}$$

with $\breve{g}_{ab}$ and $\bar{g}_{ab}$ being the components of the metric tensor in the material cordinate system $\xi$ and coordinate system $x$ connected with the undeformed rod, respectivelly,[2]

$$\bar{E}^{abcd} = \lambda\bar{g}^{ab}\bar{g}^{cd} + \mu\bar{g}^{ac}\bar{g}^{bd} + \mu\bar{g}^{ad}\bar{g}^{bc} \tag{6}$$

and

$$dV = |\breve{g}_{ab}|^{1/2}\,\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3. \tag{7}$$

Nevertheless, we consider incompressible material; then $|\breve{g}_{ab}|^{1/2} = 1$ and $\mathrm{d}V = \mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3$.

We denote the operators of the partial derivatives with respect to $\xi^i$ as

$$\partial_i\bullet \equiv \bullet_{,i} \equiv \frac{\partial\bullet}{\partial\xi^i} \quad , \quad \partial_{ij}\bullet \equiv \bullet_{,ij} \equiv \frac{\partial^2\bullet}{\partial\xi^i\partial\xi^j}. \tag{8}$$

Let's now analyze the positional relations within an element of the deformed rod, see Figure 2. For the direction of the tangent of the centreline at the point $A$ it holds

$$\sin\psi = \frac{\partial x_A^2}{\partial\xi^1} \equiv \partial_1 x^2 \quad , \quad \cos\psi = \frac{\partial x_A^1}{\xi^1} \equiv \partial_1 x^1 \tag{9}$$

and for the distance $\overline{A_oB}$ we can write (omitting the index B: $\xi^2 \equiv \xi_B^2$)

$$\begin{pmatrix} 0 \\ \xi^2 \end{pmatrix} + \mathbf{u} = \mathbf{v} + \xi^2\begin{pmatrix} -\partial_1 x^2 \\ \partial_1 x^1 \end{pmatrix}. \tag{10}$$

[2]Generally, $\breve{T}^{pq\ldots}_{ab\ldots}$: components of a tensor $T$ in curvilinear coordinate system $\xi$, $\bar{T}^{pq\ldots}_{ab\ldots}$: components of a tensor $T$ in Cartesian coordinate system $x$.

The position of a point of the deformed rod we express as

$$\begin{aligned} \text{centreline point}: \quad & \mathbf{x}_A = \begin{pmatrix} \xi^1 \\ 0 \end{pmatrix} + \mathbf{v} \\ \text{general point}: \quad & x_B^i \equiv x^i = \xi^i + u^i, \end{aligned} \tag{11}$$

where $x^i$ are its coordinates in the Cartesian system $x$ connected with the undeformed rod, $\xi^i$ are its coordinates in the material system $\xi$, and $u$ and $v$ are the components of the displacement vector of the corresponding points. Combining (10) and (11), and according to Figure 2, we can write

$$\mathbf{u} = \mathbf{v} + \xi^2\begin{pmatrix} -\partial_1 v^2 \\ \partial_1 v^1 \end{pmatrix} \equiv \mathbf{v} + \xi^2\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{P}}\partial_1\mathbf{v}. \tag{12}$$

Together with the relations

$$\begin{aligned} \bar{g}_{ab} &= \delta_a^{\ c}\delta_b^{\ d}\bar{g}_{cd} \\ \breve{g}_{ab} &= \partial_a x^c\partial_b x^d\bar{g}_{cd} \\ &= \delta_{ab} + \delta_{ad}\partial_b u^d + \delta_{bc}\partial_a u^c \end{aligned} \tag{13}$$

among the metric tensors $\bar{g}_{ab}$ and $\breve{g}_{ab}$ of the coordinate systems "$x$" and "$\xi$", respectivelly, [$\delta_i^j \equiv 1\,(i=j)$; $\delta_i^j \equiv 0\,(i \neq j)$: the Kronecker's unit tensor (Kronecker's delta)] derived utilizing (11), and neglecting the term $\partial_a u^c\partial_b u^c$ (Bernoulli hypothesis for a bended beam[3]) it is possible to rewrite (5) as

$$2\bar{\gamma}_{ab} = \partial_b u_a + \partial_a u_b. \tag{14}$$

In the system $x$ we consider the vanishing of the Christoffel's symbols and the equality of covariant and contravariant vector components ($\bullet_a = \bullet^a$) and by substituting (6) and (14) into (4) and after some modifications we arrive at

$$\bar{E}^{abcd}\bar{\gamma}_{ab}\bar{\gamma}_{cd} = \lambda(\partial_a u_a)^2 + \mu(\partial_a u_b\partial_a u_b + \partial_a u_b\partial_b u_a). \tag{15}$$

Now, if we approximate the longitudinal and transversal displacements $v^1$ and $v^2$ by the Fourier series with the base functions according to (2) and (3), respectively, i.e., in the matrix form

$$\begin{aligned} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} &= \begin{pmatrix} K_1^1\ K_2^1\ \ldots & \mathbf{0} \\ \mathbf{0} & K_1^2\ K_2^2\ \ldots \end{pmatrix} \begin{pmatrix} V_1^1 \\ V_2^1 \\ \vdots \\ V_1^2 \\ V_2^2 \\ \vdots \end{pmatrix} \\ &\equiv \begin{pmatrix} \mathbf{K}^1 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^2 \end{pmatrix}\begin{pmatrix} \mathbf{V}^1 \\ \mathbf{V}^2 \end{pmatrix} \equiv \mathbf{KV}, \end{aligned} \tag{16}$$

we can finally, substituting $(11)_2$ into (10), solving it for $\mathbf{u}$, and substituting into (15), arrive at[4]

$$\begin{aligned} \bar{E}^{abcd}\bar{\gamma}_{ab}\bar{\gamma}_{cd} = \mathbf{V}^T & \begin{pmatrix} \mathbf{K}_{,1} + \xi^2\mathbf{PK}_{,11} \\ \mathbf{PK}_{,1} \end{pmatrix}^T (\lambda\mathbf{P}_0 + \\ & + \mu\mathbf{E} + \mu\mathbf{P}_1)\begin{pmatrix} \mathbf{K}_{,1} + \xi^2\mathbf{PK}_{,11} \\ \mathbf{PK}_{,1} \end{pmatrix}\mathbf{V}, \end{aligned} \tag{17}$$

[3]I.e., all the rod's cross-sections are planar and perpendicular to the deformed centreline of the rod; that is why it is natural to use orthogonal curvilinear coordinate system.

[4]$\mathbf{K}_{,1} \equiv \dfrac{\partial\mathbf{K}}{\partial\xi^1}$, as stated in (8).

where

$$\mathbf{P}_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad , \qquad \mathbf{P}_1 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

and $\mathbf{E}$ is the unit matrix of type (4×4).

Then, the total elastic energy of the rod given by integration

$$U = \frac{1}{2}\int_0^l \int_{-h}^h \bar{E}^{abcd}\bar{\gamma}_{ab}\bar{\gamma}_{cd}\mathrm{d}\xi^2\mathrm{d}\xi^1 w \tag{19}$$

is obtained symbolically using the Maxima software.[5]

## V. Kinetic energy

The kinetic energy of an element of the rod (in a matrix form) is

$$T = \frac{1}{2}\int_V \rho\dot{\mathbf{b}}^T\dot{\mathbf{b}}\mathrm{d}V. \tag{20}$$

Indeed, for the position of the element we have [using $(11)_2$, (12), and (16)]

$$\mathbf{b} = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \mathbf{A}\mathbf{x} = \mathbf{A}\left(\xi + \underbrace{(\mathbf{K} + \xi^2\mathbf{P}\mathbf{K}_{,1})}_{\mathbf{B}}\mathbf{V}\right) \tag{21}$$
$$\mathbf{A} \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

and for its time derivative

$$\begin{aligned}\dot{\mathbf{b}} &= \dot{\mathbf{A}}(\xi + \mathbf{B}\mathbf{V}) + \mathbf{A}\mathbf{B}\dot{\mathbf{V}} \\ &= \dot{\varphi}\mathbf{A}_{,\varphi}(\xi + \mathbf{B}\mathbf{V}) + \mathbf{A}\mathbf{B}\dot{\mathbf{V}} \\ &= \begin{pmatrix} \mathbf{A}\mathbf{B} & \mathbf{A}_{,\varphi}(\xi + \mathbf{B}\mathbf{V}) \end{pmatrix}\begin{pmatrix} \dot{\mathbf{V}} \\ \dot{\varphi} \end{pmatrix}.\end{aligned} \tag{22}$$

One could cast the mass matrix in the form

$$\mathcal{M} \equiv \begin{pmatrix} \mathbf{B}^{\mathbf{T}}\mathbf{B} & \mathbf{B}^{\mathbf{T}}\mathbf{P}^{\mathbf{T}}(\xi + \mathbf{B}\mathbf{V}) \\ sym. & (\xi + \mathbf{B}\mathbf{V})^T(\xi + \mathbf{B}\mathbf{V}) \end{pmatrix} \tag{23}$$

but this is unnecessary using Maxima that can perform the integration of $(\dot{\mathbf{b}}, \dot{\mathbf{b}})$ directly:

$$T = \frac{1}{2}\rho w \int_0^l \int_{-h}^h \dot{\mathbf{b}}^T\dot{\mathbf{b}}\mathrm{d}\xi^2\mathrm{d}\xi^1. \tag{24}$$

## VI. Potential energy of external forces

The potential energy of the gravitational force field reads

$$\begin{aligned}Z &= \textstyle\int_0^l w \int_{-h}^h b^2\rho g\mathrm{d}\xi^2\mathrm{d}\xi^1 \\ &= \rho g w \begin{pmatrix} \sin\varphi & \cos\varphi \end{pmatrix}\left(\begin{pmatrix} hl^2 \\ 0 \end{pmatrix} + 2h\textstyle\int_0^l \mathbf{K}\mathrm{d}\xi^1\mathbf{V}\right)\end{aligned} \tag{25}$$

whilst the work of the external moment, $M(t)$, applied at the point $O$ is

$$W = \int_{\psi_0}^{\psi} M(t)\mathrm{d}\psi, \tag{26}$$

where

$$\psi = \varphi + \mathtt{atan2}(v^2_{,1}, v^1_{,1}) \tag{27}$$

considering $\varphi$ being the rotation of the system $x$ and $v^2_{,1}/v^1_{,1}$ being the tangent of the deformed rod at the point $\xi^1 = 0$ in the coordinate system $x$.

[5]http://maxima.sourceforge.net/

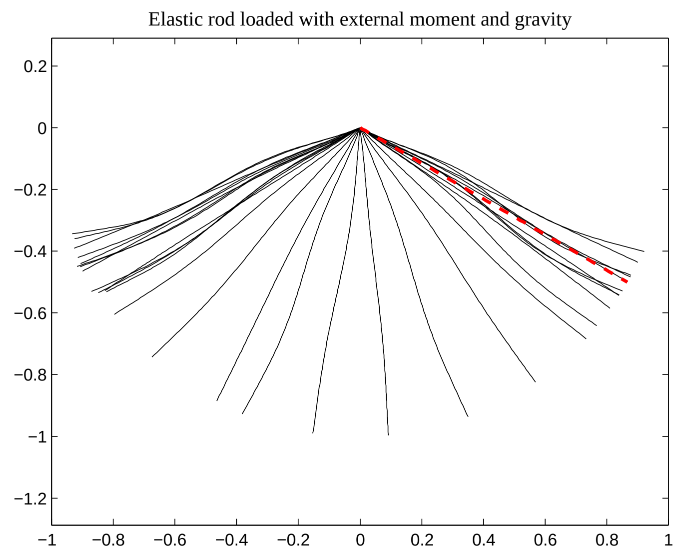


Fig. 3. The animation of the rotating rod during 1.8s with deformation zoom 300x.

## VII. Equations of motion

The necessary condition of the minimum of action gives the equations of motion

$$\left(\frac{\partial L}{\partial \dot{V}}\right)^{\bullet} - \frac{\partial L}{\partial V} = 0 \quad , \qquad \left(\frac{\partial L}{\partial \dot{\varphi}}\right)^{\bullet} - \frac{\partial L}{\partial \varphi} = 0 \tag{28}$$

with the Lagrangean function $L$ (as already indicated above)

$$L \equiv T(\mathbf{V}, \dot{\mathbf{V}}) - U(\mathbf{V}) - Z(\mathbf{V}, \varphi) + M\varphi \tag{29}$$

where, apparently,

$$v^1_{,1} = \mathbf{K}^1_{,1}\mathbf{V}^1 \quad , \qquad v^2_{,1} = \mathbf{K}^2_{,1}\mathbf{V}^2. \tag{30}$$

Now, the Maxima can be advantegously employed to set the equations of motion in symbolical form.

## VIII. Solution

The equations of motion created by Maxima have been transformed, using GNU Awk (Gawk),[6] into the syntax of GNU Octave.[7]

In Figure 3 we present the simulation results of our problem for the following constants and initial condifions:

*material:*
$\mu \doteq 8.1 \cdot 10^4$ MPa
$\lambda \doteq 1.2 \cdot 10^5$ MPa
$\rho \doteq 7800\ \mathrm{kg/m^3}$

*initial conditions:*
$\varphi(0) = -\pi/6$
$\dot{\varphi}(0) = 0$
$\mathbf{V}(0) = \mathbf{0}$ (undeformed rod)
$\dot{\mathbf{V}}(0) = \mathbf{0}$

*external forces:*
$t \in (0; 0.3)$ s : $M(t) = 300 \cdot t$ N

[6]http://www.gnu.org/software/gawk/
[7]http://www.gnu.org/software/octave/

$t \geq 0.3 \text{ s} : M(t) = 0 \text{ N}$
$g = 9.81 \text{ m} \cdot \text{s}^{-2}$

*Fourier approximation:*
number of Fourier series expansion coefficients considered: 7 (for longitudinal as well as transversal deformations)

## IX. Conclusion

During the analysis we only assume small deformations (i.e., we use linear elasticity and Bernoulli hypothesis) and we do not consider energy dissipation. Further, we take into account only the finite number of elements of the Fourier series expansion and we are aware of the numerical error of the integration procedure.

We apply our procedure on an example of elastic rod rotating in a gravitational field about a fixed point and loaded with external moment.

Our approach can be used straightforwardly for the analysis of curved rods and/or rods on curved surfaces. Using the Lagrange's multipliers method the decsribed approach is applicable to the deformable multibody systems.

The analysis as well as the numerical integration has been performed using only the GPL licensed free (GPL) software packages.

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