Circular Rings for Students

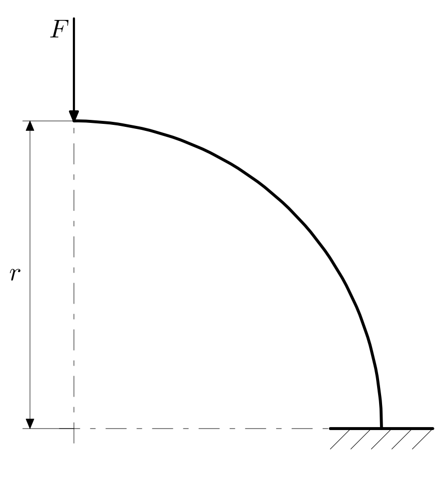
Author: Tomáš Mareš

Department of Mechanics, Faculty of Mechanical Engineering, Czech Technical University in Prague, Czech Republic, Technická 4, 166 07 Prague

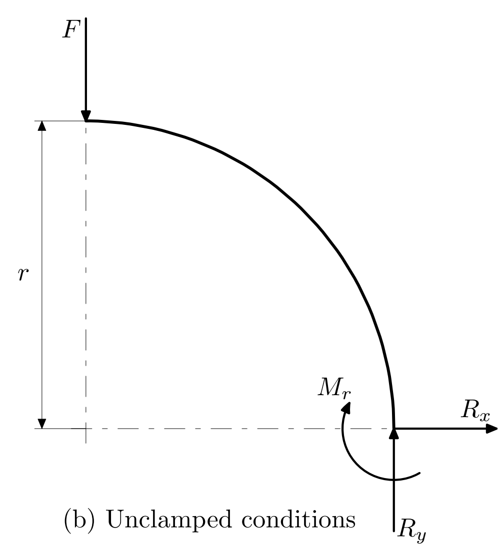
<Tomas.Mares@fs.cvut.cz>

Problem No. 1

The structure is in the form of one quadrant of a thin circular ring of radius r. One end is clamped and the other end is loaded by a vertical force F as depict at Fig. 31a. Determine the vertical displacement under the point of application of the force F. Consider only strain energy of bending.



(a) Given conditions



(b) Unclamped conditions

Obrázek 31: The thin quarter circular ring of radius r

1. Find reactions!

As a first step at dealing with problems of this type it is necessary to attempt to find reactions. For this purpose we must take supports from their places and instead of them to put reactions, see Fig. 31b. From this figure we can readily write force equilibrium equations in the horizontal direction, the vertical direction, and moment equilibrium equation with respect to build-in point, respectively:

$$R_x = 0,$$
$$R_y - F = 0,$$
$$F \cdot r - M_r = 0.$$

These are three linearly independent equations for three unknowns. Therefore it is possible to resolve this system of equations as follows.

$$R_x = 0,$$
$$R_y = F,$$
$$M_r = F \cdot r.$$

2. Decide whether the given problem is either statically determinate or statically indeterminate!

From prior item it is clear that there is no difficulty in determination of reactions from equilibrium equations and thus this problem is statically determined. This is important for we may use Castigliano's theorem.

3. Use Castigliano's theorem!

Castigliano's theorem states that the deflection v_F at the action point of force F and in the direction of this force is given by

$$v_F = \frac{\partial U}{\partial F}, \tag{86}$$

where

$$U = \int_l \frac{M^2 \mathrm{d}s}{2EI} \tag{87}$$

is elastic pontial energy (also called strain energy).

It is possible to adapt this expresion for our purposes: Substituting formulation (87) into expresion (86) we get

$$v_F = \frac{\partial}{\partial F} \int_l \frac{M^2 \mathrm{d}s}{2EI},$$

and interchanging order of derivative and integration, which is possible provided that integrand and its derivative are continuous, we arrive at

$$v_F = \int_l \frac{\partial}{\partial F} \left(\frac{M^2}{2EI} \right) \mathrm{d}s,$$

and

$$v_F = \int_l \frac{M \frac{\partial M}{\partial F}}{EI} \mathrm{d}s,$$

wherein we denote

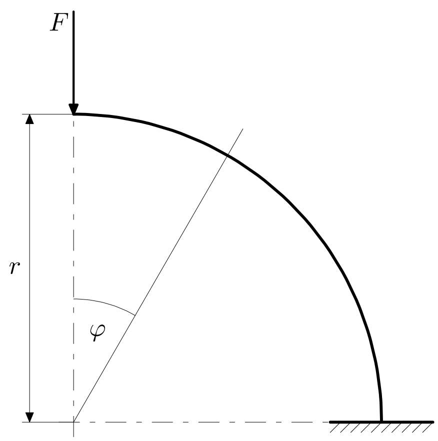
$$m = \frac{\partial M}{\partial F}.$$

As we will soon see this function m represents bending moment at points of our quarter-circle ring when this is loaded with force of magnitude 1 (without units) in the point of searched displacement and in the direction of this displacement. This force is often called the dummy unit force.

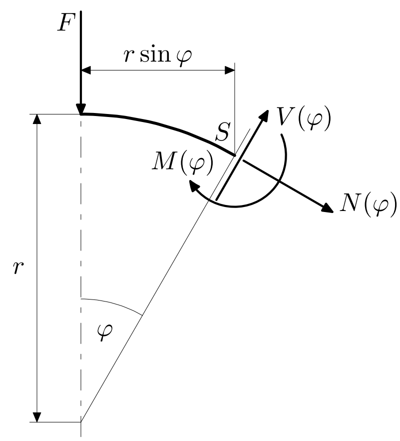
Hence we have obtained so-called Mohr's integral

$$v_F = \int_l \frac{Mm}{EI}\,\mathrm{d}s.$$

4. Use Mohr's integral



(a) Angle φ determining site of section



(b) Method of section

Obrázek 32: The assessment of internal bending moment M

Firstly, we must assess internal bending moment M as a function of some appropriate variable. It is apparent, see Fig. 34a, that such a variable is angle φ. For determination of internal bending moment we shall use the well-known method of section; cf. Fig. 34b where, utilizing moment statical equation with respect to point of section S, we obtain

$$M = Fr\sin\varphi.$$

Secondly, it is necessary to specify the derivative labelled as m. It holds that

$$m = \frac{\partial M}{\partial F},$$

and thus

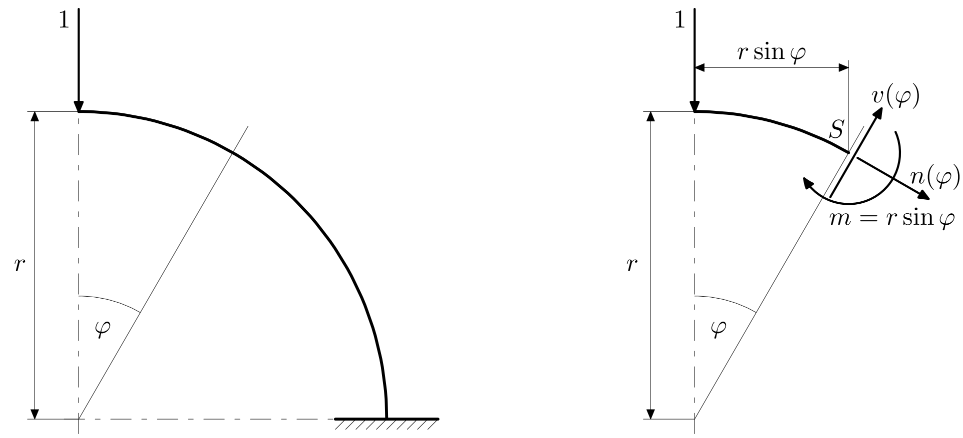
$$m = \frac{\partial(Fr\sin\varphi)}{\partial F},$$

or

$$m = r\sin\varphi.$$

It is evident that if we applied a dummy unit force at site and direction of searched deflection we would assess an internal bending moment caused by this force that would be the same as above.[118]

[118]Cf. Fig. 33.



(a) Given conditions (b) Method of section

Obrázek 33: The thin circular ring loaded with dummy unit force

This fact is very important in the cases when there is not a force acting at the spot and at the direction of searched displacement. In those cases we shall not assess the function m by using derivative of M but by evaluating internal bending moment caused with dummy unit force.

Finally, we will integrate the Mohr's integral (note that $\mathrm{d}s = r\mathrm{d}\varphi$):

$$v_F = \int_l \frac{Mm}{EI}\,\mathrm{d}s = \int_0^{\frac{\pi}{2}} \frac{Fr\sin\varphi \cdot r\sin\varphi}{EI}\,r\mathrm{d}\varphi = \frac{\pi F r^3}{4EI}.$$

Problem No. 2

Consider the structure from Problem No. 1 under the same loading, see Fig. 31a. Determine the horizontal displacement of the point where the force F is applied. Consider only strain energy of bending.

1. Find reactions!

This item is entirely identical as a item 1 of the mentioned Problem No. 1.[119] Consequently we have

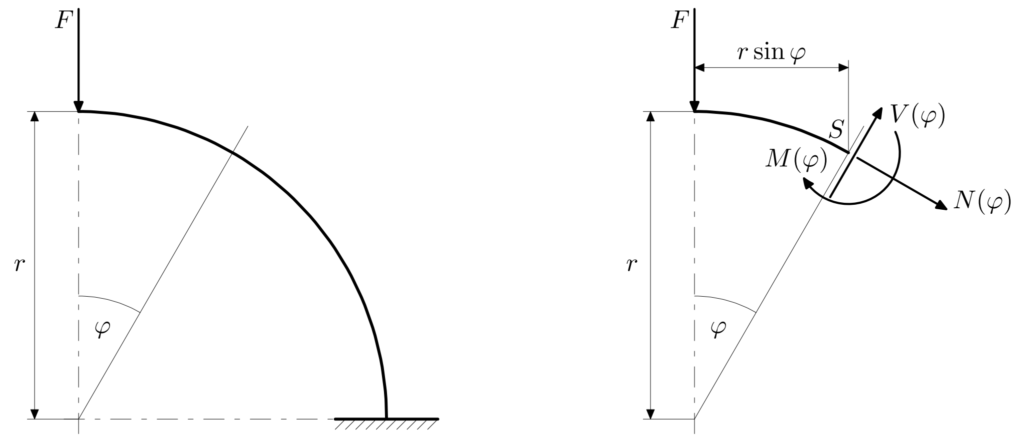
$$R_x = 0,$$

$$R_y = F,$$

$$M_r = F \cdot r.$$

2. Decide whether given problem is either statically determinate or statically indeterminate!

There was no difficulty in determination of reactions from equilibrium equations and thus this problem is statically determined. Therefore we can use Castigliano's theorem, and we use it in the form of Mohr's integral.



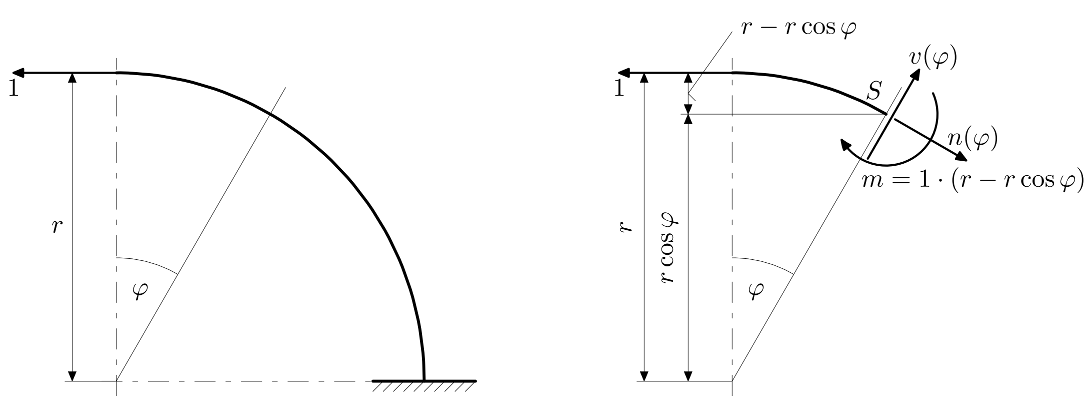
(a) Angle φ determining site of section (b) Method of section

Obrázek 34: The assessment of internal bending moment M

3. Use Mohr's integral

The assessment of internal bending moment M as a function of the appropriate variable was accomplished at fourth item of the Problem No. 1. on page 135 in the form (cf. Fig. 34b)

$$M = Fr\sin\varphi.$$



(a) Given conditions (b) Method of section

Obrázek 35: The thin circular ring loaded with dummy unit force

Now, it is necessary to specify the derivative labelled as m. In this pursuit we may apply a dummy unit force at site and direction of searched deflection. Using the method of sections we have in accordance with Fig. 35b the following expression:

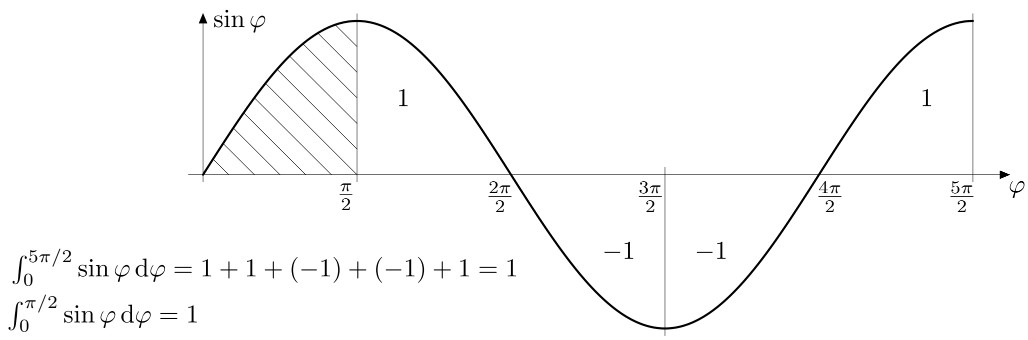
$$m = r - r\cos\varphi.$$

Finally, we shall integrate the Mohr's integral (note again that $\mathrm{d}s = r\mathrm{d}\varphi$):

$$v = \int_l \frac{Mm}{EI}\,\mathrm{d}s = \int_0^{\frac{\pi}{2}} \frac{1}{EI}\,Fr\sin\varphi\cdot(r - r\cos\varphi)\,r\mathrm{d}\varphi = \int_0^{\frac{\pi}{2}} \frac{Fr^3}{EI}\,\sin\varphi\,\mathrm{d}\varphi - \int_0^{\frac{\pi}{2}} \frac{Fr^3}{EI}\,\sin\varphi\cos\varphi\,\mathrm{d}\varphi.$$

[119]Cf. p. 134.

The first integral we can integrate with usage of geometrical explication of definite integral. We know that hatched space bounded with sine at Fig. 36 has unit area; above φ axis we consider this area with positive sign and below φ axis with negative sign.



Obrázek 36: The thin circular ring loaded with dummy unit force

The second integral

$$\int_0^{\frac{\pi}{2}} \sin\varphi \cos\varphi \, \mathrm{d}\varphi$$

is solvable through substitution

$$x = \sin\varphi;$$

then there is

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}\varphi}\,\mathrm{d}\varphi = \cos\varphi \, \mathrm{d}\varphi$$

and the integral takes form

$$\int_0^1 x \, \mathrm{d}x = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}.$$

Consequently,

$$v = \frac{Fr^3}{EI}\left(1 - \frac{1}{2}\right) = \frac{Fr^3}{2EI}.$$

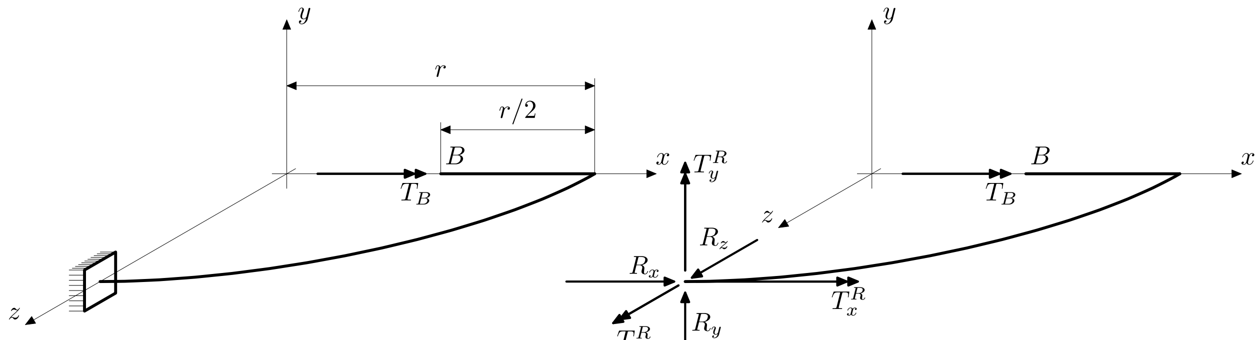
Problem No. 3

A thin circular ring of radius r in the form of one quadrant of a circle lies in x–z plane. It is fixed to the wall at one end and to the other end a straight bar with length of $0.5r$, lying also in x–z plane, is rigidly attached. Both the ring and the bar have bending rigidity EI and torsional rigidity GJ. The unsupported end is loaded by a moment, named T_B, whose vector is directed parallel to the x-axis. To emphasize that vector T_B at Fig. 37a represents a moment it is furnished with double arrow. The aim is to determine the displacement v_B of the free end, in the Fig. 37a marked as B, in the direction of y-axis.

1. Find reactions and decide whether given problem is either statically determinate or statically indeterminate!

As was said, a first step at dealing with problems of this type is to attempt to find reactions. For this purpose we must replace the supports by relevant reactions, see Fig. 37b. From this figure we can readily write force equilibrium equations in directions of all axes, and moment equilibrium equations with respect to spinning about all three axes, respectively:

$$R_x = 0,$$



(a) Given conditions (b) Unclamped conditions

Obrázek 37: The thin quarter circular ring of radius r lying in x–z plane

$$R_y = 0,$$
$$R_z = 0,$$
$$T_x^R + T_B = 0,$$
$$T_y^R = 0,$$

and at last

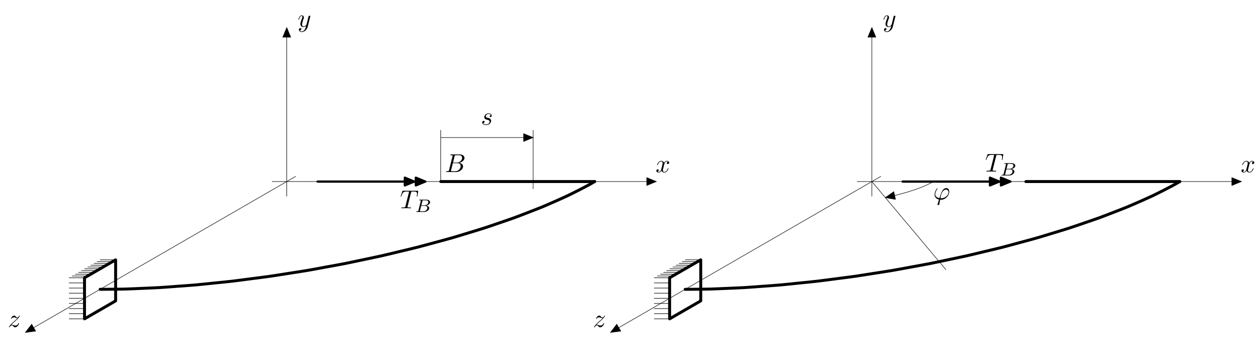
$$T_z^R = 0.$$

From these we readily have found the conclusion, that our problem is statically determinate with only one nozero reaction, whose magnitude is

$$T_x^R = -T_B.$$

2. Use Castigliano's theorem at arrangement of Mohr's integral!

The inference of the above item enables us to use Mohr's integral.

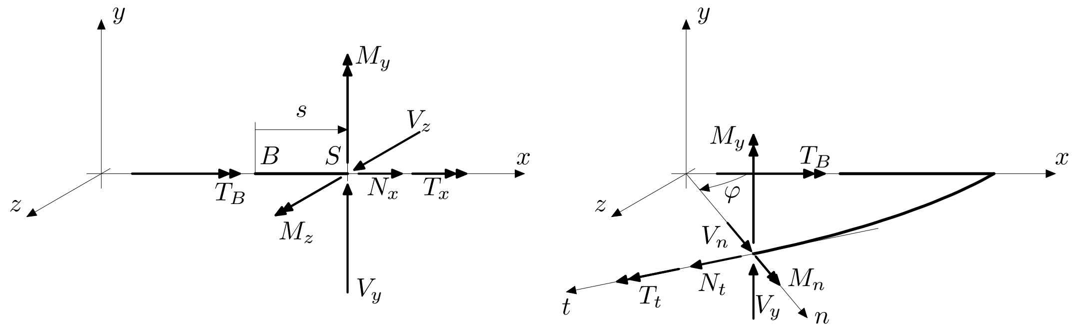


(a) Co-ordinate s in the straight part (b) Angle φ in the curved part

Obrázek 38: Choice of variables

Firstly, we must assess internal impacts awakened with moment T_B as a function of some appropriate variable. It is apparent, see Fig. 38, that such variables are co-ordinate s in the straight bar and angle φ in the curved part. The internal impacts we shall determine by method of section according to Fig. 39. It is apparent that equilibrium equations of the cut out straight part run: in the case of force equilibrium at direction of x-axis

$$N_x = 0,$$



(a) The straight part – called I. part

(b) The curved part – called II. part

Obrázek 39: The assessment of internal impacts

in the case of force equilibrium at direction of y-axis

$$V_y = 0,$$

in the case of force equilibrium at direction of z-axis

$$V_z = 0,$$

in the case of moment equilibrium with respect to x-axis

$$T_B + T_x = 0,$$

in the case of moment equilibrium with respect to y-axis

$$M_y = 0,$$

and in the case of moment equilibrium with respect to z-axis

$$M_z = 0.$$

For the curved part it reads: in the case of force equilibrium at direction of tangent t

$$N_t = 0,$$

in the case of force equilibrium at direction of y-axis

$$V_y = 0,$$

in the case of force equilibrium at direction of normal n

$$V_n = 0,$$

in the case of moment equilibrium with respect to tangent t

$$T_t - T_B \cdot \sin\varphi = 0,$$

in the case of moment equilibrium with respect to y-axis

$$M_y = 0,$$

and in the case of moment equilibrium with respect to normal n

$$M_n + T_B \cdot \cos\varphi = 0.$$

From above we see that the only nonzero internal impacts are torque moments at the straight part and the curved part, respectively

$$T_x = -T_B,$$

$$T_t = T_B \cdot \sin\varphi,$$

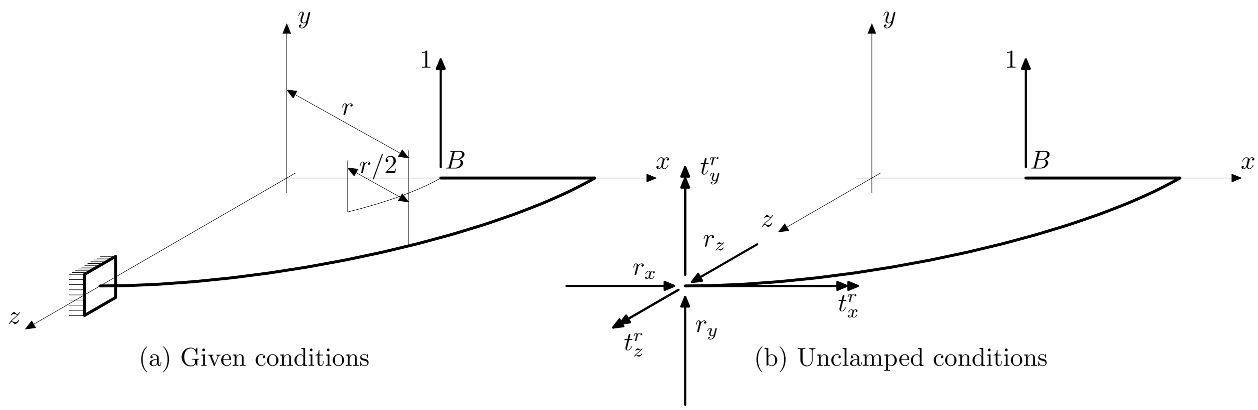
and the bending moment about normal n at the curved part

$$M_n = -T_B \cdot \cos\varphi.$$

At the case of these impacts the Mohr's integral reads

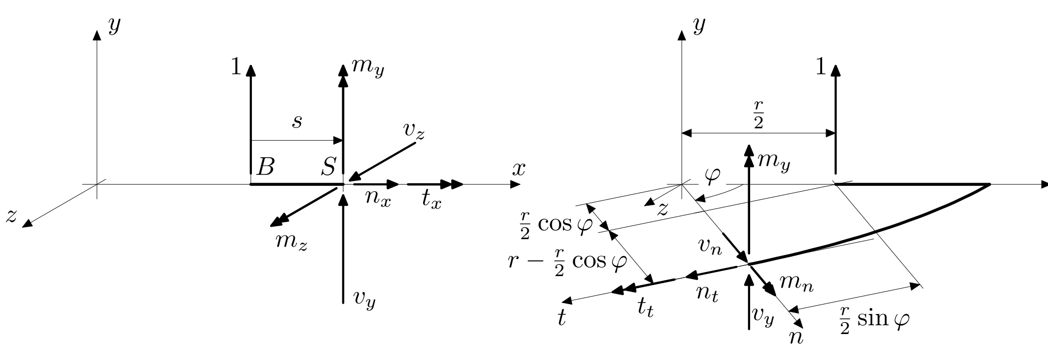
$$v_B = \int_0^{\frac{R}{2}} \frac{T_x t_x}{GJ}\,\mathrm{d}s + \int_0^{\frac{\pi}{2}} \frac{M_n m_n}{EI}\,r\mathrm{d}\varphi + \int_0^{\frac{\pi}{2}} \frac{T_t t_t}{GJ}\,r\mathrm{d}\varphi, \tag{88}$$

where m_n, t_x, and t_t are internal bending moment and internal torque moments, respectively, that are of the same kind as nonzero real internal impacts established above, produced with dummy unit force applied at the site and in the direction of searched displacement, see Fig. 40.



(a) Given conditions (b) Unclamped conditions

Obrázek 40: The thin ring loaded with dummy unit force



(a) The straight part – called I. part (b) The curved part – called II. part

Obrázek 41: The internal forces and moments inside ring loaded with dummy unit force

We may again determine them by the section method again. It holds, as is seen from Fig. 41, that equilibrium equations have successive forms: In the case of force equilibrium of the cut out straight part at direction of x-axis

$$n_x = 0,$$

at direction of y-axis

$$v_y + 1 = 0,$$

and at direction of z-axis

$$v_z = 0.$$

In the case of moment equilibrium with respect to x-axis

$$t_x = 0,$$

with respect to y-axis

$$m_y = 0,$$

and in the case of moment equilibrium with respect to z-axis

$$m_z - 1 \cdot s = 0.$$

For the curved part it reads: In the case of force equilibrium at direction of tangent t

$$n_t = 0,$$

at direction of y-axis

$$v_y + 1 = 0,$$

and in the case of force equilibrium at direction of normal n

$$v_n = 0.$$

In the case of moment equilibrium with respect to tangent t

$$t_t - 1 \cdot \left(r - \frac{r}{2} \cos\varphi \right) = 0,$$

with respect to y-axis

$$m_y = 0,$$

and finally in the case of moment equilibrium with respect to normal n

$$m_n + 1 \cdot \frac{r}{2} \sin\varphi = 0.$$

From here we have the searched quantities

$$t_x = 0,$$

$$m_n = -\frac{r}{2} \sin\varphi,$$

and

$$t_t = r - \frac{r}{2} \cos\varphi.$$

Substituting these expressions into Mohr's integral (88) we obtain

$$v_B = \frac{1}{EI} \int_0^{\frac{\pi}{2}} T_B \cos\varphi \cdot \frac{r}{2} \sin\varphi \cdot r \mathrm{d}\varphi + \frac{1}{GJ} \int_0^{\frac{\pi}{2}} T_B \sin\varphi \cdot \left(r - \frac{r}{2} \cos\varphi \right) r \mathrm{d}\varphi$$

or, after same arrangement,

$$v_B = \frac{T_B r^2}{2EI} \int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi \, \mathrm{d}\varphi + \frac{T_B r^2}{GJ} \int_0^{\frac{\pi}{2}} \sin\varphi \, \mathrm{d}\varphi - \frac{T_B r^2}{2GJ} \int_0^{\frac{\pi}{2}} \sin\varphi \cos\varphi \, \mathrm{d}\varphi$$

Using conclusions from p. 138 regarding integration antecedent integrals we arrive at

$$v = T_B r^2 \left(\frac{1}{4EI} + \frac{3}{4GJ} \right).$$