Fibre composite optimization

Tomas Mares

***Abstract*—Fibre composites are steadily becoming more and more popular both in industry and science. This progress is unvarying for decades. The call for optimization and tailoring of the properties of the fibre composites is still clamorous and the author of the lecture is going to present his humble contribution to this field of knowledge.**

The paper is devoted to an optimization, namely to the stiffness maximization, of fibre composites. The fibre composite is a composite composed of a fibre (or fibres) and matrix (or matrices). The point is that the fibre can be oriented or even bent and twisted in a suitable way.

The paper starts with a definition of the measure of stiffness as a quantity to be maximized. Then there is excursion into the stiffness maximization of thin plates. In the following, the emphasis is on curved fibres and structures; as the curved shapes are more natural for fibre composites. There arise the necessity to use the curvilinear tensor calculus. For the consistence of the used symbology the section shortly describing tensor calculus is placed at the begining of the paper. The implementation of the tensor calculus into the problems of fibre composite stiffness maximization is demonstrated on stiffness maximization of a thick walled elliptic tube. The paper is concluded with views on the future research that is based on the free fibre composite optimization.

***Index Terms*—fibre, laminate, composite, optimization, maximization, stiffness**

I. Introduction

At this paper on fibre composite optimization the author would like to present a method to optimize, in a way, fibre composites. It is well known that to optimize something without specifying the aim of the optimization is meaningless. To avoid this let us state the aim first. We want to maximize stiffness. There might be a good reason in this, but the reason is always subject to an individual (or individuals) in the problem involved. Another good aim (or object) could be the condition of constant stress state in the construction. To cheer us up we may say that results of optimization with respect to these respective objects generally leads to the same design.[1]

As the results are (if possible) the same and the formulation of the problem is much easier at the case of stiffness maximization, the author have decided to pursue the stiffness maximization problem.

II. Stiffness maximization

Let us start with a one-dimensional spring, Fig. 1, and look at the connection of a force, displacement, stiffness and work done by external forces.

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[1][All02]

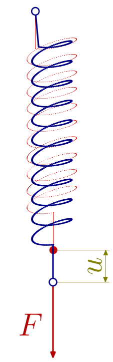

Fig. 1. The spring

For a given force, F, both the displacement

$$u = \frac{F}{k}$$

and the work done

$$W = Fu = \frac{F^2}{k}$$

is inversely proportional to the stiffness constant, k. Thus, regarding the force F as a given constant, we have the equivalence of the problem to maximize stiffness and to minimize the work done by external forces

$$\arg\max k = \arg\min Fu.$$

Accordingly, we can formulate the problem to maximize stiffness as

$$\min(f, \hat{u}),$$

where $(.,.)$ stands for the inner product, f for external forces and $\hat{u}$ for the displacement in the state of an elastic equilibrium, *i.e.*, a solution to the Navier-Cauchy equations

$$A\hat{u} = f \tag{1}$$

as well as a minimizer of the total potential energy:

$$\Pi(\hat{u}) = \min_{u \in \mathbf{U}} \Pi(u), \quad \Pi = \frac{1}{2}(Au, u) - (f, u), \tag{2}$$

$\mathbf{U}$ being a set of the statically admissible displacements. Combining (1) stated in the form

$$(A\hat{u}, \hat{u}) = (f, \hat{u})$$

with (2) yields

$$\Pi(\hat{u}) = \frac{1}{2}(A\hat{u}, \hat{u}) - (f, \hat{u}) = -\frac{1}{2}(f, \hat{u}),$$

which means that the solution of the stiffness maximization problem fulfil the following (α being design parameters)[2]

$$\hat{\alpha} = \arg\min_{\alpha}(f, \hat{u}) = \arg\max_{\alpha}\Pi(\hat{u}),$$

i.e.

$$\hat{\alpha} = \arg\max_{\alpha}\min_{u\in U}\Pi.$$

To incorporate constraints, $g = 0$, constituting the set $\mathbf{U}$ and a set $\mathbf{A}$, of design possibilities, we can build up Lagrangian $L = \Pi + \lambda g$ and write down necessary conditions of the extreme:

$$\frac{\delta L}{\delta u} = 0,$$

$$\frac{\delta L}{\delta \alpha} = 0.$$

These equations are generally nonlinear and not easy to solve. To solve these equations there is a method called alternating fulfilment of necessary conditions based on the following algorithm:

1) Deliberately choose the design variables, α_0
2) Using α_k solve the elasticity problem, $\frac{\delta L}{\delta u} = 0 \Rightarrow u_k$
3) Using u_k solve the optimum condition, $\frac{\delta L}{\delta \alpha} = 0 \Rightarrow \alpha_{k+1}$
4) If $\alpha_k = \alpha_{k+1}$ you have a solution, otherwise goto item 2)

Unfortunately, this approach does not always converge, but somewhat lengthy manipulation[3] leads to another formulation of the problem

$$\hat{\alpha} = \arg\min_{\alpha}\min_{\sigma\in C}\frac{1}{2}\int_{\Omega} C_{abcd}\sigma^{ab}\sigma^{cd}\,\mathrm{d}\Omega,$$

where C is a set of statically admissible stresses, that is valid only at the case of homogeneous kinematic boundary conditions, but with much better convergence properties.[4]

The last expression is written in the index notation of tensor calculus, a tool necessary to solve geometrically more complicated problems. As it is we should spend at least a few words on the curvilinear elasticity, *i.e.*, the theory of elasticity written in the language of (curvilinear) tensor calculus.

III. Curvilinear elasticity

To solve real life problems we certainly must take into account curved shapes and curvilinear elasticity. First, the elasticity. Elasticity is a branch of physics which studies the properties of materials that are deformed under stress (or, say, external forces), but then, when the stress is removed, return to its original shape. The amount of deformation is specified with strain.[5] The concept of elasticity is build on the classical works of Sir William Petty (London, 1674) and Robert Hooke (London, 1678/1660), and the state of an elastic body is characterized via stress and strain tensors.[6] As it is, we must take a glance on the tesor calculus[7] and its most important tensor – the metric tensor,

$$g_{ab} = \frac{\partial\theta}{\partial\xi^a}\cdot\frac{\partial\theta}{\partial\xi^b},$$

θ being the radius vector of a point of the elastic body and ξ^a a curvilinear coordinate system.[8] The contravariant metric tensor is defined as[9]

$$g^{ab} = (g_{ab})^{-1} \tag{3}$$

and the derivative of a vector as ($\boldsymbol{g}_b = \frac{\partial\theta}{\partial\xi^b}$ being a vector base)

$$\frac{\partial\boldsymbol{a}}{\partial x^a} = \nabla_a a^b\boldsymbol{g}_b,$$

with covariant derivative

$$\nabla_a a^b = \partial_a a^b + \Gamma^b_{ac}a^c,$$

where the Christoffel symbols of the second kind

$$\Gamma^d_{ab} = g^{dc}\frac{1}{2}(\partial_b g_{ac} + \partial_a g_{cb} - \partial_c g_{ab}),\ \partial_a = \frac{\partial}{\partial x^a}.$$

The well known differential operators are expressed as[10]

$$\operatorname{grad}\varphi = \nabla_a\varphi\,\boldsymbol{g}^a = \partial_a\varphi\,\boldsymbol{g}^a,\ \operatorname{div}\boldsymbol{v} = \nabla\boldsymbol{v} = \nabla_a v^a,$$

$$\operatorname{rot}\boldsymbol{A} = \nabla\times\boldsymbol{A} = \epsilon^{abc}\nabla_a A_b\,\boldsymbol{g}_c,\ \nabla^2\varphi = \operatorname{div}\operatorname{grad}\varphi.$$

Let us state the definition of the Green-Lagrange-St. Venant strain[11]

$$\overset{o}{E}_{ab} = \frac{1}{2}(\overset{\xi}{g}_{ab} - \overset{o}{g}_{ab}),$$

where $\overset{\xi}{g}_{ab}$ is a metric of the material coordinate system coincident before the deformation with the space coordinate system $\overset{o}{g}_{ab}$.

For small deformations the Green-Lagrange-St. Venant strain takes the form of the small strain tensor[12]

$$\varepsilon_{ab} = \frac{1}{2}(\overset{\xi}{g}_{ab} - \overset{o}{g}_{ab})\Big|_{\text{lin.}} = \frac{1}{2}(\nabla_a u_b + \nabla_b u_a).$$

It became commonly known and used[13] that the real state of a deformed body, $\hat{u}_a$, minimizes the total potential energy

$$\Pi(u_a) = a(u_a) - l(u_a)$$

on a set of the admissible states, $\mathbf{U}$, where the elastic strain energy

$$a(u_a) = \frac{1}{2}\int_{\Omega} E^{abcd}\varepsilon_{ab}(u_a)\varepsilon_{cd}(u_a)\mathrm{d}\Omega$$

and the potential energy of the applied forces

$$-l(u_a) = -\int_{\Omega} p^a u_a\mathrm{d}\Omega - \int_{\partial_t\Omega} t^a u_a\mathrm{d}\Gamma.$$

[2] [Ben95]
[3] [Mar06]
[4] [All02]
[5] [Cau27]
[6] [Lov27]
[7] [SS78], [LR89] and [Cia05]
[8] [GZ54]
[9] [SS78], [LR89]
[10] [LR89], [GZ54], [Wal84], [Wan05]
[11] [Ant05], [GZ54], [Cia05]
[12] [Lov27], [Was75], [Wal84]
[13] [Mik64], [Was75], [LR89], [Dac89], [Che00]. The origin of these principles is joined with such names as Maupertuis, 1746, Euler, 1744, and Lagrange, 1788.

Shortly, it holds

$$\hat{u}_a = \arg \min_{u_b \in \mathbf{U}} \Pi(u_c).$$

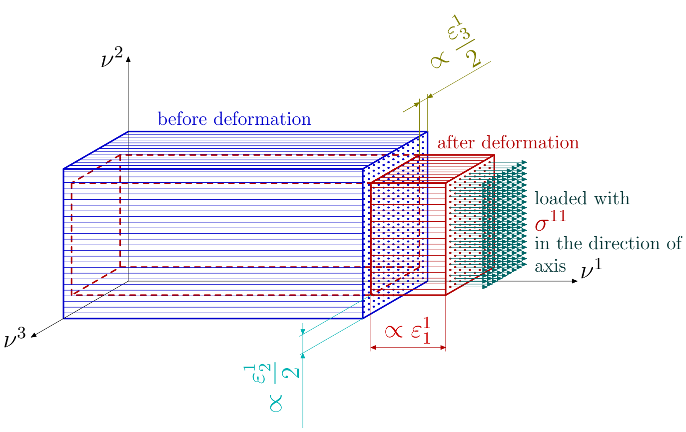

Fig. 2. Orthotropic block

In the case of orthotropic material, for example orthotropic elementary block (signed as νth block, as outlined in Figure 2) we may choose the coordinate system, ν_a, called the principal material frame. The *principal* stands for *alined with* the principal material axes of the orthotropic material. Then the elasticity tensor has the following entries,

$$\left\{ \overset{\nu}{E}{}^{abcd} \right\}_{ab\lceil cd} =$$

$$\begin{pmatrix} \Phi_{11} & 0 & 0 & 0 & \Phi_{12} & 0 & 0 & 0 & \Phi_{13} \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ \Phi_{21} & 0 & 0 & 0 & \Phi_{22} & 0 & 0 & 0 & \Phi_{23} \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ \Phi_{31} & 0 & 0 & 0 & \Phi_{32} & 0 & 0 & 0 & \Phi_{33} \end{pmatrix}$$

where

$$\Phi_{11} = \frac{1-\nu_{23}\nu_{32}}{N} E_1, \quad \Phi_{12} = \frac{\nu_{21}+\nu_{23}\nu_{31}}{N} E_1,$$

$$\Phi_{13} = \frac{\nu_{31}+\nu_{32}\nu_{21}}{N} E_1, \quad \Phi_{21} = \frac{\nu_{12}+\nu_{13}\nu_{32}}{N} E_2,$$

$$\Phi_{22} = \frac{1-\nu_{13}\nu_{31}}{N} E_2, \quad \Phi_{23} = \frac{\nu_{32}+\nu_{31}\nu_{12}}{N} E_2,$$

$$\Phi_{31} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{N} E_3, \quad \Phi_{32} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{N} E_3,$$

$$\Phi_{33} = \frac{1-\nu_{12}\nu_{21}}{N} E_3,$$

and

$$N = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{32}\nu_{21},$$

where E_k and ν_{kl} are the appropriate Young's moduluses and Poisson ratios, respectively. The ν above the tensor symbol indicates that the symbol does not symbolize an abstract tensor but that it stands for the tensor components in the ν-frame of reference and $\{ab\lceil cd\}$ indicates how the entries are stored in the array, namely that the rows belong successively to the following pairs of indices ($ab = 11, 12, 13, 21, 22, 23, 31, 32, 33$) and the columns to the couples ($cd = 11, 12, \ldots, 33$).

The above relations may be readily used in a very large variety of anisotropic materials via the concept of locally orthotropic material.

The concept of locally orthotropic material is based on the thought that at every point of a material it is possible to construct a cartesian coordinate system ν_a such that the material in (infinitesimal) surrounding behaves orthotropically, *i.e.*, the mentioned relations hold.

Thus we only need to perform a transformation from the principal frame of orthotropy, ν^a, into a frame of the computation. It must be said that in the frame of the computation, the tensor entries are not necessarily physical quantities.[14]

IV. The simplest (illustrating) problem of fibre composite stiffness maximization

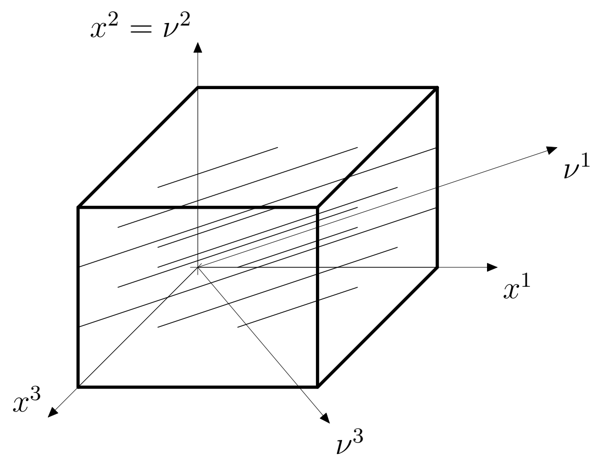

Fig. 3. Fibre composite

To illustrate the process of optimizing the fibre angle orientation, that may be continually changing along the length of the fibre, let us suppose the block from Fig. 3, tensioned with uniform stress, σ, in the direction of the x^1 axis, *i.e.*, the stress tensor of the body is

$$\sigma_{ab} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which solves the step 2) of the algorithm. The material properties are orthogonal with principal axes of orthotropy ν^a. In the coordinate system ν^a the following relation of stress to strain holds

$$\overset{\nu}{\varepsilon}_{ab} = \overset{\nu}{C}_{abcd} \overset{\nu}{\sigma}{}^{cd},$$

where the compliance tensor

$$\left\{ \overset{\nu}{C}_{abcd} \right\}_{\{ab\lceil cd\}} =$$

[14] [MD06], [Mar07a]

$$\begin{matrix} \frac{1}{E_{11}} & 0 & 0 & 0 & -\frac{\nu_{21}}{E_{22}} & 0 & 0 & 0 & -\frac{\nu_{31}}{E_{33}} \\ 0 & \frac{1}{4G_{12}} & 0 & \frac{1}{4G_{12}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 \\ 0 & \frac{1}{4G_{12}} & 0 & \frac{1}{4G_{12}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & 0 & 0 & 0 & \frac{1}{E_{22}} & 0 & 0 & 0 & -\frac{\nu_{32}}{E_{33}} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & \frac{1}{4G_{23}} & 0 \\ 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & \frac{1}{4G_{23}} & 0 \\ -\frac{\nu_{13}}{E_{11}} & 0 & 0 & 0 & -\frac{\nu_{23}}{E_{22}} & 0 & 0 & 0 & \frac{1}{E_{33}} \end{matrix}$$

But now, in the x coordinate system, the stiffness maximization problem of ours may be written

$$\min_{\alpha} \int_V c \, \mathrm{d}V,$$

where

$$c = \overset{x}{\sigma}{}^{ab} \overset{x}{C}_{abcd} \overset{x}{\sigma}{}^{cd} = \sigma \, \overset{x}{C}_{1111} \, \sigma.$$

Using the transformation rule

$$\overset{x}{C}_{abcd} = \frac{\partial \nu^i}{\partial x^a} \frac{\partial \nu^j}{\partial x^b} \frac{\partial \nu^k}{\partial x^c} \frac{\partial \nu^l}{\partial x^d} \, \overset{\nu}{C}_{ijkl},$$

i.e.,

$$\overset{x}{C}_{1111} = \frac{\partial \nu^i}{\partial x^1} \frac{\partial \nu^j}{\partial x^1} \frac{\partial \nu^k}{\partial x^1} \frac{\partial \nu^l}{\partial x^1} \, \overset{\nu}{C}_{ijkl},$$

$$\overset{x}{C}_{1111} = \begin{pmatrix} \cos^2\alpha & 0 & \cos\alpha\sin\alpha & 0 & 0 & 0 & \sin\alpha\cos\alpha & 0 & \sin^2\alpha \end{pmatrix} \times$$

$$\times \left\{ \overset{\nu}{C}_{abcd} \right\}_{\{ab\lceil cd\}} \begin{pmatrix} \cos^2\alpha \\ 0 \\ \cos\alpha\sin\alpha \\ 0 \\ 0 \\ 0 \\ \sin\alpha\cos\alpha \\ 0 \\ \sin^2\alpha \end{pmatrix},$$

α being the angle contained between axes x^1 and ν^1. Thus,

$$c = \sigma^2 \left(\cos^4\alpha \frac{1}{E_{11}} + \cos^2\alpha \sin^2\alpha \left(\frac{1}{G_{13}} - \frac{\nu_{31}}{E_{11}} - \frac{\nu_{13}}{E_{33}} \right) + \sin^4\alpha \frac{1}{E_{33}} \right).$$

The necessary condition

$$\frac{\partial c}{\partial \alpha} = 0$$

reads

$$\cos^3\alpha \sin\alpha A_1 + \cos\alpha \sin^3\alpha A_2 = 0,$$

with

$$A_1 = \frac{1}{G_{13}} - \frac{\nu_{31}}{E_{11}} - \frac{\nu_{13}}{E_{33}} - \frac{2}{E_{11}}$$

and

$$A_2 = \frac{2}{E_{22}} + \frac{\nu_{31}}{E_{11}} + \frac{\nu_{13}}{E_{33}} - \frac{1}{G_{13}}.$$

The last condition permits the following solutions:

1) $\hat{\alpha}_1 = \pm\frac{\pi}{2}$ with $c_1 = \frac{\sigma^2}{E_{33}}$
2) $\hat{\alpha}_2 = 0, \pi$ with $c_2 = \frac{\sigma^2}{E_{11}}$
3) $\hat{\alpha}_{3,4} = \arctan\left(\pm\sqrt{-\frac{A_1}{A_2}}\right)$.

Which one of the solutions of necessary conditions is solution of the stiffness maximization problem depends on numerical values of the material characteristics and should be decided from numerical values of the object function c or from a study of the second derivative of c.[15]

So much about the philosophy of the solving procedure. Now, let us look on more practical problems.

V. Stiffness maximization of plates

Using the elasticity principles described above we can formulate the problem of stiffness maximization at the case of laminated multilayer Kirchhoff plates of symetric layout in symbolic form as[16]

1) The elasticity problem

$$P^{abcd} w_{cd} = q^{ab}$$

2) The necessary condition of optimum

$$w_{ab} w_{cd} R^{abcd}(\alpha_\nu) = 0$$

where w_{ab} represents Fourier series expansion coefficients of the perpendicular displacement and $R^{abcd}(\alpha_\nu)$ are functions of the design parameters, α_ν, standing for the layer orientation, see Fig. 4. The loading is expanded into Fourier series with coefficients, q^{ab}.

There is only space for citing a few results of the described problem in this paper. The results of stiffness maximization of the laminate plate are quoted in Figs 4 through 8 where there are descriptions of the loading conditions in the caption of the figures and the optimal angles of layer orientations in the figures (only one half of the symmetric plates is depicted).[17]

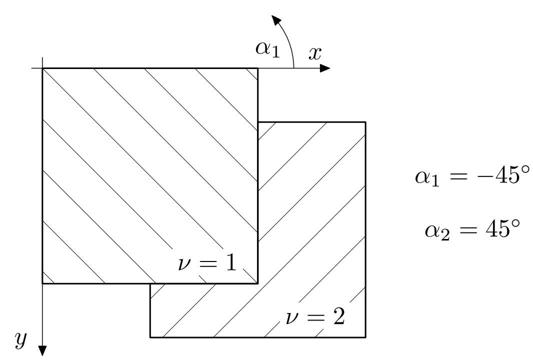

Fig. 4. Square plate (a=b being a length of the sides) of four layers loaded by $q = q_o \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$ (Permutation of layout is possible)

VI. Stiffness maximization of thick-walled anisotropic elliptic tube

At this chapter we are going to maximize the stiffness, by choosing winding angle of the fibre, of a thick walled fibre wound, and thus anisotropic, elliptic tube. The tube is optimized for three different loadings. First, under a pulling force, F, see Fig. 9, distributed evenly along the lower face. Second and third, with shearing forces T_1 (a force in the direction of the axis b^1) and T_2 (having direction of the b^2), respectively. Both of the last two forces are evenly distributed as well. Following the scheme of the alternative fulfilment of necessary condition method we must first perform the analysis.

[15] [Mar05]
[16] [Mar04]
[17] [Mar06]

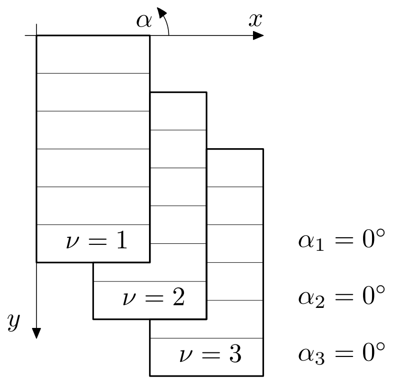

Fig. 5. Rectangular plate (a:b=1:2) of six layers loaded by $q = q_o \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$

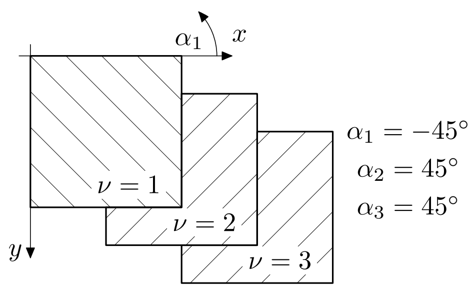

Fig. 6. Square plate (a:b=1:1) of six layers loaded by $q = q_o \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$ (All other layouts are possible as well)

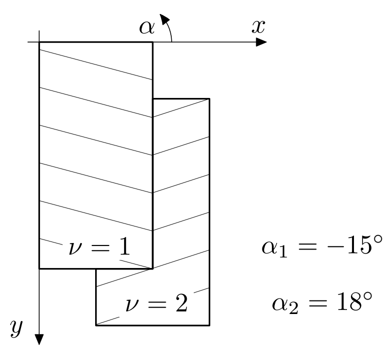

Fig. 7. Rectangular plate (a:b=1:2) of four layers loaded by $q = q_o xy$ (The inverse layout is equivalent)

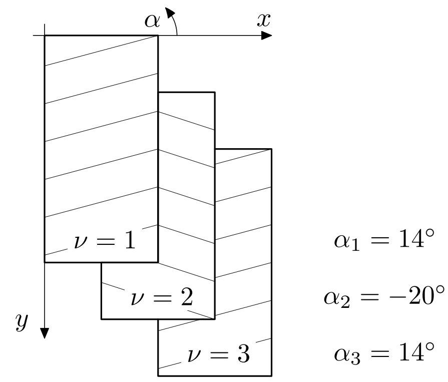

Fig. 8. Rectangular plate (a:b=1:2) of six layers loaded by $q = q_o xy$ (The other layouts are equivalent)

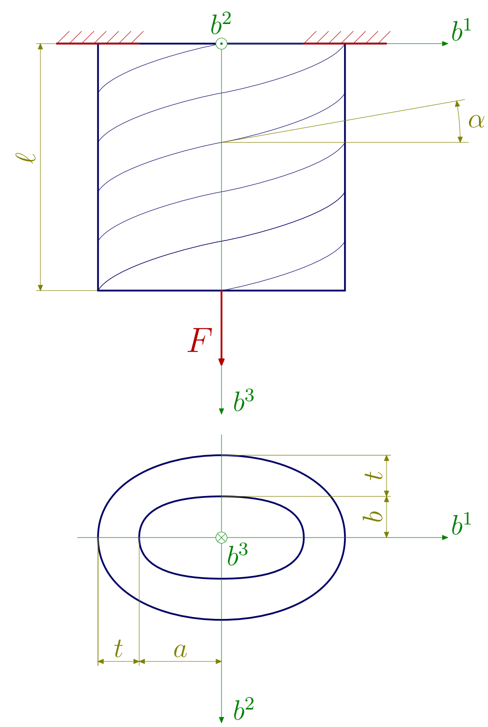

Fig. 9. The anisotropic elliptic tube

A. The analysis

In this subsection we focus on an analysis of a thick-walled elliptic tube which is wound by a fiber embedded in a matrix (Fig. 9). The upper end of the tube is clamped and a uniformly distributed force, *e.g.* F as indicated in the Fig., is applied on the lower end. The fiber is wound at an angle, α, that is going to be optimized, in the sense of maximal stiffness, in the subsequent sections. The analysis is performed using the concept of locally orthotropic material, where the elasticity tensor is expressed in a local Cartesian coordinate system aligned with the principal directions of local orthotropy of the material. A system of coordinate transformations from the local Cartesian coordinate systems into a global coordinate system of computation is performed. The total potential energy of the problem is expressed in the global coordinate system. After approximating the dependent variables, representing the displacements, by Fourier series, the total potential energy is minimized.

In the way described elsewhere,[18] the elastic energy is expressed as

$$a = \frac{1}{2}\,\mathtt{A}^T K \mathtt{A},$$

where $\mathtt{A}$ represents Fourier series coefficients and K a kind of the stiffness matrix.

The work of the applied force (for the case of loading in accord with the Fig. 9) may be expressed as

$$l = \mathtt{P}' * \mathtt{A}$$

[18] [Mar07b]

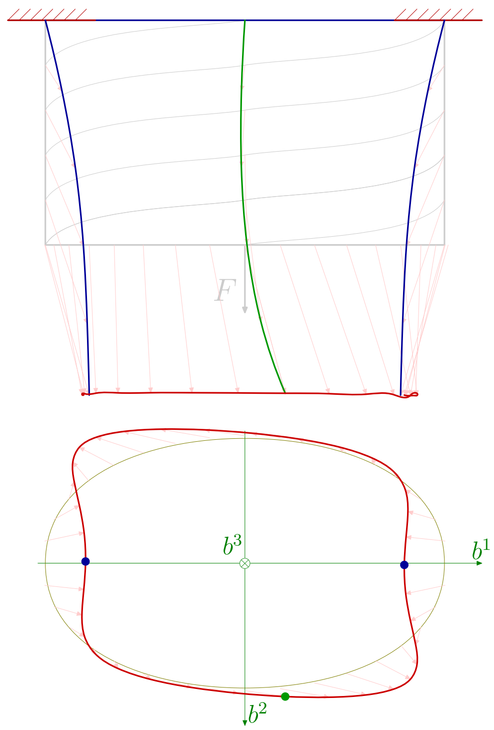

Fig. 10. Deformation of the elliptic tube

with `P` being a known vector. The resulting displacements, `ub`, in the global coordinate system, b, for the case of a given angle α, is obtained relativelly easily and the deformed shape is demonstrated in Fig. 10.

B. Stiffness maximization

Applying the method of alternating fulfilment of necessary conditions as described above leads to the necessity to solve two problems.

1) The problem of elasticity as already solved in the form

$$\mathtt{A} = \mathtt{K}^{-1}\mathtt{P}$$

2) The stiffness maximum condition,

$$\frac{\partial \Pi}{\partial \alpha} = 0,$$

i.e.,

$$\frac{1}{2}\mathtt{A}^T \frac{\partial \mathtt{K}}{\partial \alpha}\mathtt{A} = 0,$$

that represents, at the regarded case, one equation and we can solve it numerically, *e.g.*, using Bisection method.

At the last equation

$$\frac{\partial \mathtt{K}}{\partial \alpha} = \int_0^{\ell}\int_0^{2\pi}\int_0^{t} \mathtt{(B-Gam)'}*\frac{\partial \mathtt{Ex}}{\partial \alpha}*\mathtt{(B-Gam)*sqrt(det(gx))}\,\mathrm{d}^3x,$$

$$\frac{\partial \mathtt{Ex}}{\partial \alpha} = \left\{ \frac{\partial \overset{x}{E}{}^{abcd}}{\partial \alpha} \right\}_{ab\lceil cd},$$

$$\frac{\partial \overset{x}{E}{}^{abcd}}{\partial \alpha} = \left(\alpha^a_i \frac{\partial x^b}{\partial \nu^j}\frac{\partial x^c}{\partial \nu^k}\frac{\partial x^d}{\partial \nu^l} + \frac{\partial x^a}{\partial \nu^i}\alpha^b_j\frac{\partial x^c}{\partial \nu^k}\frac{\partial x^d}{\partial \nu^l} + \right.$$

$$\left. + \frac{\partial x^a}{\partial \nu^i}\frac{\partial x^b}{\partial \nu^j}\alpha^c_k\frac{\partial x^d}{\partial \nu^l} + \frac{\partial x^a}{\partial \nu^i}\frac{\partial x^b}{\partial \nu^j}\frac{\partial x^c}{\partial \nu^k}\alpha^d_l \right) \overset{\nu}{E}{}^{ijkl},$$

where

$$\alpha^a_b = \frac{\partial}{\partial \alpha}\left[\frac{\partial x^a}{\partial \nu^b}\right] = \frac{\partial x^a}{\partial b^c}\frac{\partial b^c}{\partial \xi^d}\frac{\partial}{\partial \alpha}\left[\frac{\partial \xi^d}{\partial \nu^b}\right],$$

$$\frac{\partial}{\partial \alpha}\left[\frac{\partial \xi^d}{\partial \nu^b}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\alpha & -\cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \end{pmatrix}.$$

The last equations are stated only as a demonstration of the simplicity of the approach. For full understanding of the symbols, one must look at more detailed description of the preceding analysis.[19]

C. Optimized winding angles

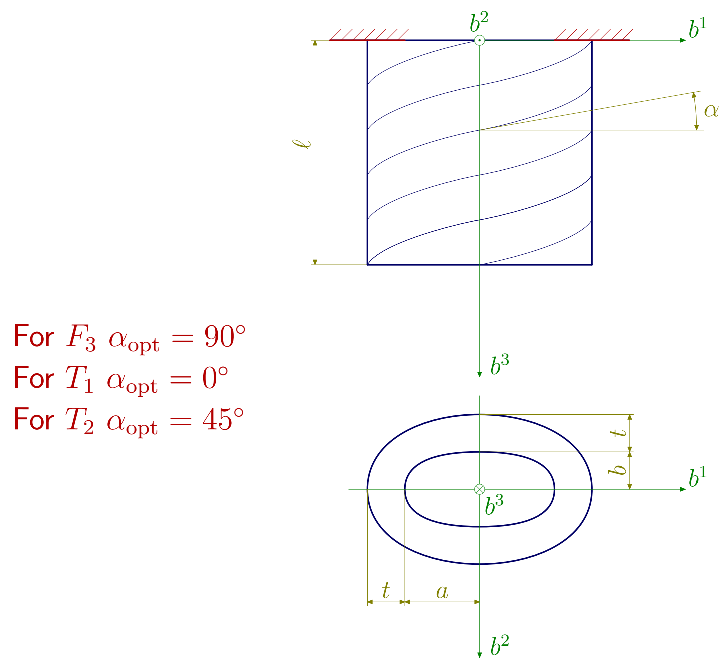

Fig. 11. The optimum angle α for given loadings

Using the above described procedure we arrive at the following results. At the case of the pulling force, F_3, the stiffness maximizing angle seems to be $90°$. For the shearing force T_1 the angle is $0°$ and for the shearing force T_2 it is $\pm 45°$, see Fig. 11. Even at this simple one parametric case, there is necessity at every step to choose the appropriate solution of the equation from the second step of the algorithm (as there is more than one solution of this equation.)

VII. Concluding Remarks

As the problem of the last section is a one-parametric one, and thus an easy one, we can check the results by direct evaluation of the objective function in a set of discrete points. It must be stated the computation time was similar at both cases. The results were essentially the same. But, of

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course, the method of alternative fulfilment of the necessary condition is of universal usage even at the case of multilayer (multiparameter) problems. The pointwise method must be at such a case substituted by another method, *e.g.*, Genetic Algorithms. As shown in the case of plates and tubes,[20] such an approach costs much more computation time and increase of uncertainty of the quality of the solution.

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